Smoothness Theorems for Generalized Symmetric Pollaczek Weights on $(-1,1)$

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Abstract

In this note a new characterisation of smoothness is obtained for weighted polynomial approximation in $L_p(1 \le p \le \infty)$ with repect to a large class of exponential weights in $(-1,1)$ which include the classical Pollaczek weights. Along the way we prove Marchaud inequalities, saturation theorems, existence theorems for derivatives and generalize a theorem of D. Lubinsky.

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1 Introduction

The purpose of this note, is to generalize [5, (1.22)] of Lubinsky for a large class of symmetric exponential weights on $(-1,1)$ which include the classical Pollaczek weights and thus extend earlier results of Ditzian and Totik for Jacobi type weight functions. Along the way, we prove Marchaud inequalities, saturation and quasi r monotonicity theorems, existence theorems for derivatives and clarify a statement made by Lubinsky in [6, Section 5, pg 19].

In [5], Lubinsky has recently investigated forward and converse theorems of polynomial approximation in $L_p(1 \le p \le \infty)$ for a class of symmetric non Szegő weights in $(-1,1)$. By a symmetric non Szegő weight in $(-1,1)$, we mean a weight

$$w := \exp(-Q)$$

where $Q : (-1,1) \to \mathbb{R}$ is even and unlike classical Jacobi weights, vanishes so strongly near ± 1 that it violates the classical Szegő condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx > -\infty.$$

As examples we mention

$$w(x) = w_{0,\alpha}(x) := \exp\left(-(1-x^2)^{-\alpha}\right), \, \alpha > 0, x \in (-1,1)$$

and

$$w(x) = w_{k,\alpha}(x) := \exp(-\exp_k([1-x^2]^{-\alpha})), \alpha > 0, k \geq 1, x \in (-1,1).$$

Here $\exp_k(;) := \exp(\exp(...(\exp(;))))$ denotes the kth iterated exponential and $\exp_0(x) = x$. In particular, $w_{0,1/2}$ is the well known Pollaczek weight, see [9, (1.10), pg 389].

For such weights, w, we define the error of best weighted approximation by

$$E_n[f]_{w,p} := \inf_{P \in \Pi_n} \|(f-P)w\|_{L_p(-1,1)}, f \in L_{p,w}(-1,1) \tag{1.1}$$

where Π_n denotes the class of polynomials of degree at most $n \geq 1$,

$$L_{p,w}(-1,1) := \{f : (-1,1) \longrightarrow \mathbb{R} : fw \in L_p(-1,1), 1 \leq p \leq \infty\}$$

and if $p = \infty$, f is further continuous and satisfies

$$\lim_{|x| \to 1} fw(x) = 0.$$

It is well known, see [6], that

$$E_n[f]_{w,p} \to 0, \, n \to \infty.$$

We denote by $P^*_{n,p} = P^*_n$, the best approximating polynomial at which the infimum in (1.1) is attained.

For parameters $r > 1$, $0 < \alpha < r$, $1 \leq p \leq \infty$, a modulus of smoothness $\omega_{r,p}(f, w, ;)$ which is defined in (1.4) below and for positive constants C_j, $j = 1, 2$ independent of n and t, Lubinsky in [5, (1.22)] established the equivalence:

(a)

$$E_n[f]_{w,p} \leq C_1 n^{-\alpha}, \, n \to \infty. \tag{1.2}$$

(b)

$$\omega_{r,p}(f, w, t) \leq C_2 t^{\alpha}, \, t \to 0^+. \tag{1.3}$$

The importance of the above result lies in the fact that it enables one to say something about the 'smoothness' properties of the function f knowing in advance how fast one can approximate it by weighted polynomials and visa versa.

Indeed will show the following:

(1) For $0 < \alpha \le r$ and for rates of decrease as fast as a negative power of n, there exists a new and more optimal characterisation of smoothness which implies (1.2) and is equivalent to (1.3).

(2) The importance of this new characterisation lies in the often different behaviour of the rth and $(r+1)$th moduli of smoothness and to this end we completely describe this relationship by proving a Marchaud inequality and a corresponding converse theorem which works in $L_p(0 < p \le \infty)$.

(3) A closer examination of (1.2) and (1.3) reveals that they also hold for a certain logarithmic rate of decrease ψ slower than a negative power of n and that for such ψ, the characterisation (1.2) and (1.3) is more optimal so that in general both characterisations are applicable to different ranges and supplement each other.

(4) A saturation theorem, quasi r monotonicity theorem and existence theorem for derivatives hold true and in the process we clarify a statement of Lubinsky in [6, Section 5, pg 19] and extend results of Ditzian and Totik, see [4, Chapter 7,8].

To state our main results, we define formally our class of weights, our modulus of smoothness and introduce some needed notation.

First we say that a real valued function $f : (a, b) \longrightarrow (0, \infty)$ is *quasi increasing (quasi decreasing)* if there exists a positive constant C_1 such that

$$a < x < y < b \Longrightarrow f(x) \le C_1 f(y) \, (f(x) \ge C_1 f(y)).$$

For any two sequences (b_n) and (c_n) of nonzero real numbers, we write

$$b_n \;\, . \;\, c_n,$$

if there exists a constant $C_1 > 0$, independent of n such that

$$b_n \le C_1 c_n$$

and

$$b_n \;\sim\; c_n,$$

if there exist positive constants C_j, $j = 2, 3$, independent of n such that

$$C_3 c_n \leq b_n \leq C_2 c_n.$$

Similar notation will be used for functions and sequences of functions. By C we will mean a positive absolute constant which may take on different values in different places.

Our weight class, much as in [5], is defined as follows:

Definition 1.1

Let

$$w = \exp(-Q)$$

where $Q : (-1, 1) \longrightarrow \mathbb{R}$ *is even, continuous, has limit* ∞ *at 1 and* Q' *is positive in* $(0, 1)$. *Then we shall write* $w \in \mathcal{E}$ *if the following conditions below hold.*

(a) $xQ'(x)$ *is strictly increasing in* $(0, 1)$ *with right limit* 0 *at* 0.

(b)

$$T(x) := \frac{Q'(x)}{Q(x)}$$

is quasi increasing in $(C_1, 1)$ *for some* $0 < C_1 < 1$.

(c) *Assume that for each* $\varepsilon > 0$, *there exist constants* $C_j > 0, j = 1, 2$ *such that uniformly for* x *and* y

$$\frac{Q'(y)}{Q'(x)} \leq C_1 \left(\frac{Q(y)}{Q(x)} \right)^{1+\varepsilon}, y \geq x \geq C_2.$$

(d) *For some* $\delta > 0$ *and* $0 < C_1 < 1$, $(1 - x^2)^{1+\delta} Q'(x)$ *is increasing in* $(C_1, 1)$.

In particular, $w_{0,\alpha}$ and $w_{k,\alpha} \in \mathcal{E}$.

In [5], the following modulus of continuity was studied for the class $\mathcal{E}$.

Definition 1.2

Let $w \in \mathcal{E}$, $0 < p \leq \infty$, $f \in L_{p,w}(-1, 1)$, $r \geq 1$, $t \in (0, t_0)$ *and set:*

$$\begin{aligned} \omega_{r,p}(f, w, t) : &= \sup_{0 < h \leq t} \|\Delta^r_{h\Phi_t(x)}(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/2t})} \\ &+ \inf_{R \in \Pi_{r-1}} \|(f - R)w\|_{L_p(|x| \geq a_{1/4t})}. \end{aligned} \qquad (1.4)$$

Here,

$$\begin{aligned}\Phi_t(x) : &= \left|1 - \frac{|x|}{a_{1/t}}\right|^{\frac{1}{2}} + T(a_{1/t})^{\frac{-1}{2}}, x \in (-1,1), &(1.5)\\ \Delta_h^r(f, x, (-1,1)) : &= \begin{cases} \sum_{i=0}^{r} \binom{r}{i}(-1)^i f(x + \frac{rh}{2} - ih) & , x \pm \frac{rh}{2} \in (-1,1) \\ 0 & , \text{otherwise} \end{cases}\end{aligned}$$

is the rth **symmetric difference** *of f and $a_{1/t}$ is the $(1/t)$th* **Mhaskar-Rakhmanov-Saff number** *for w^2 defined by*

$$1/t = \frac{2}{\pi}\int_0^1 \frac{a_{1/t}uQ'(a_{1/t}u)}{\sqrt{1-u^2}}du.$$

For those who are unfamilar, its significance lies partly in the identity

$$\|Pw\|_{L_\infty(-1,1)} = \|Pw\|_{L_\infty(-a_n,a_n)},\ P \in \Pi_n.$$

For example for classical Jacobi weights, the interval $[-a_n, a_n]$ is essentially $[-1-n^{-2}, 1-n^{-2}]$ and thus the remaining subintervals of $[-1,1]$ of length n^{-2} are negligible. For the class of weights $\mathcal{E}$ however, a_n is much smaller and so it is more significant. For example for $w_{k,\alpha}$,

$$\frac{1-a_n}{(\log_k n)^{-1/\alpha}} \sim 1$$

where $\log_k$ denotes the usual kth iterated logarithm.

The function $h\Phi_t$ is a suitable replacement for the well known factor $h\sqrt{1-x^2}$ in the Ditzian-Totik modulus, see [4], i.e. it describes the improvement in the degree of approximation over $\{x : a_{\alpha n} \leq |x| \leq a_{\frac{n}{2}}\}$ for any fixed $\alpha \in (0, 1/2)$ in much the same way as $\sqrt{1-x^2}$ does for Jacobi weights on $[-1,1]$.

Following is our first main result:

Theorem 1.3

Let $w \in \mathcal{E}$, $0 < \alpha \leq r$, $1 \leq p \leq \infty$ and $f \in L_{p,w}(-1,1)$. Further define

$$\psi(\tau) := \tau^\alpha,\ \tau \geq 0.$$

Then

$$\omega_{r,p}(f, w, t) \lesssim t^\alpha,\ t \to 0^+ \tag{1.6}$$

iff

$$\|P_n^{*(r)} \Phi_{\frac{1}{n}}^r w\|_{L_p(-1,1)} \ . \ n^r \psi(1/n), \ n \to \infty. \tag{1.7}$$

Moreover (1.7) *implies*

$$E_n[f]_{w,p} \ . \ n^{-\alpha}, \ n \to \infty. \tag{1.8}$$

In particular it is well known, see [8], that

$$E_n[f]_{w,p} \ . \ n^{-r}, \ n \to \infty.$$

does not imply that

$$\omega_{r,p}(f, w, t) \ . \ t^r, \ t \to 0^+$$

but as we have seen

$$\|P_n^{*(r)} \Phi_{\frac{1}{n}}^r w\|_{L_p(-1,1)} \ . \ 1, \ n \to \infty$$

does.

We deduce that in the range for which $\omega_{r,p}(f, w, ;)$ and $\omega_{r+1,p}(f, w, ;)$ have different behavior, $E_n[f]_{w,p}$ yields information on $\omega_{r+1,p}(f, w, ;)$ only while

$$\|P_n^{*(j)} \Phi_{\frac{1}{n}}^j w\|_{L_p(-1,1)}$$

yields information on $\omega_{j,p}(f, w, ;)$ for $j = r$ and $j = r+1$. Concerning the precise relationship between $\omega_{r,p}(f, w, ;)$ and $\omega_{r+1,p}(f, w, ;)$ the following Marchaud inequality and corresponding converse theorem hold true.

Theorem 1.4

Let $w \in \mathcal{E}$, $0 < p \leq \infty$, $q = \min(1, p)$, $t \in (0, t_0)$ *and* $f \in L_{p,w}(-1,1)$. *Then uniformly for* f *and* t*,*

$$\omega_{r,p}(f, w, t) \quad . \quad t^r \left[\int_t^C \frac{\omega_{r+1,p}(f, w, u)^q \left(\log_2\left(\frac{1}{t}\right)\right)^{rq/2}}{u^{rq}} du \right. \tag{1.9}$$
$$\left. + \quad \left(\log_2\left(\frac{1}{tr}\right)\right)^{rq/2} \|fw\|_{L_p(-1,1)}^q \right]^{1/q}.$$

Moreover

$$\omega_{r+1,p}(f, w, t) \ . \ \omega_{r,p}(f, w, t). \tag{1.10}$$

The appearance of (1.9) might seem at first unatural because of the presence of the logarithmic terms. However they arise because of the function Φ_t in the modulus (1.4) which is necessary to depict endpoint effects in the Mhaskar-Rakhmanov-Saff interval. They also appear for Erdős weights in $\mathbb{R}$, see [2], and in earlier work of Ditzian and Totik, see [4]. The estimate (1.10) is classical and follows [4] and [1].

Theorem 1.5

Let $w \in \mathcal{E}$, $\sigma > 0$, $1 \leq p \leq \infty$ and $f \in L_{p,w}(-1,1)$. Further define for ε sufficiently small and positive

$$\psi(\tau) := (\log 1/\tau)^{-\sigma},\ \tau \in [0,\varepsilon).$$

Then

(a)

$$\omega_{r,p}(f,w,t)\ .\ \psi(t),\ t \to 0^{+} \tag{1.11}$$

iff

$$E_n[f]_{w,p}\ .\ \psi(1/n),\ n \to \infty. \tag{1.12}$$

Moreover

$$\|P_n^{*(r)}\Phi^r_{\frac{1}{n}}w\|_{L_p(-1,1)}\ .\ n^r\psi(1/n),\ n \to \infty \tag{1.13}$$

yields essentially no information on the function f.

(b) *We always have*

$$\|P_n^{*(r)}\Phi^r_{\frac{1}{n}}w\|_{L_p(-1,1)}\ .\ n^r \int_0^{1/n} \frac{\omega_{r,p}(f,w,\tau)}{\tau}d\tau. \tag{1.14}$$

We deduce that for the slow decreasing ψ as above, the characterisation (1.2) and (1.3) is better whereas for faster decreasing ψ, Theorem 1.3 is the correct replacement. Thus both theorems are applicable to different ranges and supplement each other. A similar effect occurs in the unweighted case, see [4, Theorem 7.3.2], for Jacobi type weights on $(-1,1)$, see [4, Chapter 8] and for Freud and Erdős weights on $\mathbb{R}$, see [1, Theorem 2.5] and [3, Theorem 2.2].

Concerning, (1.14) this is non trivial as the modulus in (1.4) is not necessary increasing. Nevertheless, using a strong quasi r monotonicity property of the modulus (1.4) which we will establish in Theorem 1.7 below, we are able to establish (1.14) and this may then be used to give an alternative proof of the implication :

$$\|P_n^{*(r)}\Phi^r_{\frac{1}{n}}w\|_{L_p(-1,1)}\ .\ n^{r-\alpha},\ n \to \infty.$$

if

$$\omega_{r,p}(f,w,t) \lesssim t^{\alpha},\ t \to 0^{+}$$

for $0 < \alpha \leq r$.

We pause briefly to outline the structure of this paper. In Section 2, we present the proofs of Theorems 1.3-1.4 and the implications (1.11), (1.12) and (1.13) of Theorem 1.5. In Section 3, we formulate and prove a saturation theorem, Theorem 1.6, a quasi r monotonicity theorem, Theorem 1.7 and in the process clarify a statement of Lubinsky in [6, Section 5, pg 19]. We then prove (1.14) of Theorem 1.5 and finally formulate and prove an existence theorem for derivatives, Theorem 1.8. We close with some final remarks and open problems.

2 The Proof of Theorems 1.3-1.4 and implications (1.11)-(1.13)

We begin with the:

Proof of Theorem 1.3

The proof follows from the following observation which is of independent interest.

Suppose that uniformly for $n \geq n_0$

$$\|P_n^{*(r)}\Phi^r_{\frac{1}{n}}w\|_{L_p(-1,1)} \lesssim n^r\psi\left(\frac{1}{n}\right) \tag{2.1}$$

for some quasi increasing or quasi decreasing

$$\psi : (0,\varepsilon) \longrightarrow [0,\infty)$$

satisfying

$$\psi(x) \longrightarrow 0, x \longrightarrow 0^{+}.$$

Here, ε is a sufficiently small positive number.

Then the following hold:

(a) Uniformly for $n \geq n_0$ and $t \in (0,t_0)$

(i)

$$E_n[f]_{w,p} \lesssim \left(\int_0^{1/n} \frac{\psi(\tau)}{\tau}d\tau\right). \tag{2.2}$$

(ii)

$$\omega_{r,p}(f,w,t) \; . \; \left(\int_0^{Ct} \frac{\psi(\tau)}{\tau} d\tau\right). \tag{2.3}$$

(b) In particular, if uniformly for $t \in (0, t_0)$,

$$\int_0^{Ct} \frac{\psi(\tau)}{\tau} d\tau \; . \; \psi(t) \tag{2.4}$$

then uniformly for $n \geq n_0$ and $t \in (0, t_0)$

$$E_n[f]_{w,p} \; . \; \psi\left(\frac{1}{Cn}\right) \tag{2.5}$$

and

$$\omega_{r,p}(f,w,t) \; . \; \psi(t). \tag{2.6}$$

We follow the method of [4, Theorem 7.3.2] and let $P_{2n}^*(f) = P_{2n}^*$ be the best approximant to f from Π_{2n} satisfying

$$\|(f - P_{2n}^*)w\|_{L_p(-1,1)} = E_{2n}[f]_{w,p}. \tag{2.7}$$

Moreover let $P_n^*(P_{2n}^*)$ be the best approximant to P_{2n}^* from Π_n satisfying,

$$\|(P_{2n}^* - P_n^*(P_{2n}^*))w\|_{L_p(-1,1)} = E_n[P_{2n}^*]_{w,p}. \tag{2.8}$$

First observe that using (1.1) and the fact that $P_n^*(P_{2n}^*)$ is a polynomial of degree at most n gives

$$\begin{aligned} E_n[f]_{w,p} &= \inf_{P \in \Pi_n} \|(f-P)w\|_{L_p(-1,1)} \\ &\leq \|(f - P_n^*(P_{2n}^*))w\|_{L_p(-1,1)}. \end{aligned} \tag{2.9}$$

Then (2.7) and (2.9) yield

$$\begin{aligned} I_n : &= \|(P_{2n}^* - P_n^*(P_{2n}^*))w\|_{L_p(-1,1)} \\ &\geq \|(f - P_n^*(P_{2n}^*))w\|_{L_p(-1,1)} - \|(f - P_{2n}^*)w\|_{L_p(-1,1)} \\ &\geq E_n[f]_{w,p} - E_{2n}[f]_{w,p}. \end{aligned} \tag{2.10}$$

We now need the estimates, see [5, (1.17)], [5, (5.9)], [5, (1.23)],

$$\omega_{r,p}(f,w,1/n) \sim \|(f-P_{2n}^*)w\|_{L_p(-1,1)} + \left(\frac{1}{2n}\right)^r \|P_{2n}^{*(r)}\Phi_{\frac{1}{2n}}^r w\|_{L_p(-1,1)} \quad (2.11)$$

and

$$\omega_{r,p}(P_{2n}^*,w,1/n) \;.\; n^{-r}\|{P_{2n}^*}^{(r)} w\Phi_{1/n}^r\|_{L_p(-1,1)}. \quad (2.12)$$

Combining these with (2.1) gives

$$\begin{aligned} I_n \quad & . \quad \omega_{r,p}\left(P_{2n}^*, w, \frac{1}{n}\right) \\ & . \quad \psi\left(\frac{1}{n}\right). \end{aligned} \quad (2.13)$$

(2.10) and (2.13) then readily give

$$E_n[f]_{w,p} \;.\; \sum_{k=1}^{\infty} \psi\left(\frac{1}{2^k n}\right) = S_n \quad (2.14)$$

where

$$S_n := \sum_{k=1}^{\infty} \psi\left(\frac{1}{2^k n}\right), n \geq n_0. \quad (2.15)$$

First observe that for each fixed k

$$\int_{\frac{1}{2^k n}}^{\frac{1}{2^{k-1} n}} \frac{1}{\tau} d\tau = \log 2$$

and assume without loss of generality that ψ is quasi-increasing for the other case is similar.

Then the quasi-monotonicity of ψ gives,

$$\begin{aligned} S_n \quad & . \quad \sum_{k=1}^{\infty} \int_{\frac{1}{2^k n}}^{\frac{1}{2^{k-1} n}} \frac{\psi(\tau) d\tau}{\tau} \\ & . \quad \int_0^{\frac{1}{n}} \frac{\psi(\tau)}{\tau} d\tau. \end{aligned} \quad (2.16)$$

Substituting (2.16) into (2.14) gives (2.2).

To see (2.3), we let $t \in (0, t_0)$, define n to be the largest integer $\leq 1/t$ and use (2.11) and the identity

$$\omega_{r,p}(f,w,t) \sim \omega_{r,p}(f,w,1/n) \quad (2.17)$$

which holds unformly for n.

Then (2.3) follows as in (2.2) using (1.1) and (2.1). Thus we have shown (2.2) and (2.3). Applying the claim above with $\psi(\tau) := \tau^\alpha$ shows that (1.7) implies (1.6) and (1.8). The reverse implication follows from (2.11). □

We next present the:

Proof of Theorem 1.5: (1.11)-(1.13)

We first observe that the implication (1.11) to (1.12) follows from (2.17) and the identity, see [5, (1.20)],

$$E_n[f]_{w,p} \;.\; \omega_{r,p}(f,w,1/n) \tag{2.18}$$

uniformly for the given n. Moreover, it is clear that we cannot deduce from (1.13) anything about the smoothness of the function f if we recall (2.4) and the definition of ψ. Thus it remains to prove the implication (1.12) to (1.11). To this end we choose $n \geq n_0$, set $l := \log_2 n$ and recall the identity, see [5, (1.21)]

$$\omega_{r,p}(f,w,1/n) \;.\; (1/n)^r \sum_{j=-1}^{l} (l-j+1)^{r/2} 2^{jr} E_{2^j}[f]_{w,p}. \tag{2.19}$$

From (2.19) and assuming (1.12), we obtain

$$\omega_{r,p}(f,w,1/n) \;.\; (1/n)^r \sum_{j=-1}^{l} (l-j+1)^{r/2} 2^{jr} \psi(1/2^j).$$

Then the above becomes

$$\begin{aligned} \omega_{r,p}(f,w,1/n) \quad &. \quad \psi(1/n) \sum_{j=-1}^{l} (l-j+1)^{r/2} \frac{(l/j)^\sigma}{2^{(l-j)r}} \\ &. \quad \psi(1/n). \end{aligned} \tag{2.20}$$

Now for the given $t \in (0, t_0)$, we set n to be the largest integer $\leq 1/t$. Then the implication (1.12) to (1.11) follows from (2.20) and the identity (2.17). This completes the proof of the implications (1.11)-(1.13). □.

Next we present the:

Proof of Theorem 1.4 (b)

Let $n \geq n_0$, $q = \min(1, p)$ and let P_n^* be the best approximant to f satisfying (1.1). Then it follows from (2.11), (2.18) and the Markov-Bernstein inequality, see [5, Lemma 2.3],

$$\|P_n^{*(r+1)} \Phi_{\frac{1}{n}}^{r+1} w\|_{L_p(-1,1)} \;.\; n \|P_n^{*(r)} \Phi_{\frac{1}{n}}^{r} w\|_{L_p(-1,1)} \tag{2.21}$$

that for $n \geq n_0$,

$$\omega_{r+1,p}\left(f, w, \frac{1}{n}\right)^q$$

$$. \ \|(f - P_n^*)w\|^q_{L_p(-1,1)} + \left(\frac{1}{n}\right)^{(r+1)q} \|P_n^{*(r+1)}\Phi^{r+1}_{\frac{1}{n}}w\|^q_{L_p(-1,1)}$$

$$. \ E_n[f]^q_{w,p} + \left(\frac{1}{n}\right)^{rq} \|P_n^{*(r)}\Phi^{r}_{\frac{1}{n}}w\|^q_{L_p(-1,1)}$$

$$. \ \omega_{r,p}\left(f, w, \frac{1}{n}\right)^q.$$

Finally for the given $t \in (0, t_0)$, let n be the largest integer $\leq 1/t$ and apply (2.17) and (2.21). This establishes (1.10). □

Before we may proceed with the proof of Theorem 1.4 (a) we need a lemma which generalizes [7, (7.2)] and which will prove useful in the proof of Theorem 1.6 as well.

Lemma 2.1 Let $w \in \mathcal{E}$, ε, $\alpha > 0$ and for any $y, z > 0$ set

$$\Psi_{y,z}(x) := \frac{\Phi_y(x)}{\Phi_z(x)},\ x \in (-1, 1) \tag{2.22}$$

where Φ_t is defined by (1.5). Then uniformly for $0 < s \leq t \leq t_0$,

$$\left(\log(2 + \frac{t}{s})\right)^{\frac{-\alpha}{2}} . \left(\sup_{x\in[-1,1]} (\Psi_{t,s}(x))\right)^{\alpha} . \left(\frac{t}{s}\right)^{\varepsilon}. \tag{2.23}$$

Proof

Firstly the lower bound in (2.23) follows from (7.2) of [7]. Thus it suffices to establish the corresponding upper bound. Firstly if $|x| \leq a_{1/t}$, then the result follows by (3.2) of [7] since in this case

$$\Psi_{t,s}(x) \ . \ 1$$

uniformly for s, t and x. Thus we may assume without loss of generality that $|x| > a_{1/t}$.

We first show that uniformly for t and x

$$\Phi_t(x) \ . \left|1 - \frac{|x|}{a_{1/2t}}\right|^{1/2}.$$

To see this, first observe that [5, (2.3)] implies that

$$\max\left(\left|1-\frac{|x|}{a_{1/t}}\right|^{1/2}, T(a_{1/t})^{-1/2}\right) \;.\; \left|1-\frac{|x|}{a_{1/2t}}\right|^{1/2}$$

for our range of $|x|$.

Now using the estimate above, the lower bound in (2.23), [7, (7.1)], the triangle inequality and (1.5) yields

$$\begin{aligned}
&\Phi_t(x) && (2.24)\\
&\;.\; \left|1-\frac{|x|}{a_{1/s}}\right|^{1/2} + \left|1-\frac{a_{1/s}}{a_{1/2t}}\right|^{1/2}\left[\left|1-\frac{|x|}{a_{1/s}}\right|^{1/2}+1\right]\\
&\;.\; \left[\Phi_s(x) + \left(\frac{a_{1/s}}{a_{1/2t}}\right)^{\frac{1}{2}}\left|1-\frac{a_{1/2t}}{a_{1/s}}\right|^{1/2}\Phi_s(x)\right.\\
&\left.+\left(\frac{a_{1/s}}{a_{1/2t}}\right)^{1/2}\left|1-\frac{a_{1/2t}}{a_{1/s}}\right|^{1/2} T(a_{1/2t})^{1/2}\left(\frac{T(a_{1/s})}{T(a_{1/2t})}\right)^{1/2}\Phi_s(x)\right]\\
&\;.\; \left(\frac{T(a_{1/s})}{T(a_{1/t})}\right)^{1/2}\left(\frac{a_{1/s}}{a_{1/t}}\right)^{1/2}\sqrt{\log\left(2+\frac{2t}{s}\right)}\,\Phi_s(x).
\end{aligned}$$

We now estimate each of the terms in (2.24).

Firstly as T is quasi increasing it follows from Definition 1.1 (c) and [7, (2.7)] that

$$\left(\frac{T(a_{1/s})}{T(a_{1/t})}\right)^{1/2} \;.\; \left(\frac{Q(a_{1/s})}{Q(a_{1/t})}\right)^{\varepsilon} \;.\; (t/s)^{\varepsilon}.$$

On the otherhand we always have

$$\left(\frac{a_{1/s}}{a_{1/t}}\right)^{1/2} \;.\; 1.$$

Inserting these estimates into (2.24), recalling that logarithms grow slower than any polynomial and dividing by $\Phi_s(x)$ yields the upper bound in (2.23) and hence the lemma. $\Box$

We are now ready for the:

Proof of Theorem 1.4 (a)

We let first $n \geq n_0$. Then if [;] denotes the largest integer $\leq$;, we may write using (2.11) and (2.18)

$$\omega_{r,p}\left(f, w, \frac{1}{n}\right)^q \tag{2.25}$$

$$\cdot \left[\omega_{r+1,p}\left(f,w,\frac{1}{n}\right)^q + \left(\frac{1}{n}\right)^{rq} \left\|P_n^{*(r)}\Phi_{1/n}^r w\right\|_{L_p(-1,1)}^q\right].$$

Now choose $l = l(n)$ with

$$r2^{l+2} \geq n \geq r2^{l+1}$$

and $n \geq 2r$. We then write

$$P_n^*(x) = \sum_{k=0}^{l-1}\left(P^*_{\left[\frac{n}{2^k}\right]}(x) - P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right) + P^*_{\left[\frac{n}{2^{l+1}}\right]}(x).$$

Applying (2.23), (2.21) and (1.1) yields

$$\left\|P_n^{*(r)}\Phi_{1/n}^r w\right\|_{L_p(-1,1)}^q \tag{2.26}$$

$$\cdot \sum_{k=0}^{l-1}\left(\left[\frac{n}{2^{k+1}}\right]\right)^{rq}(k+2)^{rq/2}\left\|\left(P^*_{\left[\frac{n}{2^k}\right]}(x) - P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)w\right\|_{L_p(-1,1)}^q$$
$$+\left(\left[\frac{n}{2^l}\right]\right)^{rq} l^{rq/2}\|fw\|_{L_p(-1,1)}^q.$$

For our given t, set $n = [1/t]$. Then we may, using (1.1) and (2.18), express (2.26) as an integral and combining this with (2.25) and (2.17) obtain the result. Thus Theorem 1.4 is completely proved □.

3 Saturation theorem, Quasi r monotonicity theorem, Existence theorem for derivatives and (1.14)

In this section, we establish a saturation theorem, a quasi r monotonicity theorem, and an existence theorem for derivatives which are of independent interest and arise naturally from our previous considerations. In the process, we clarify a statement raised in [6, Section 5, pg 19] and prove (1.14).

We begin with:

Theorem 1.6 *Let $w \in \mathcal{E}$, $1 \leq p \leq \infty$, $f \in L_{p,w}(-1,1)$ and $r \geq 1$. Suppose that for a given $\varepsilon > 0$,*

$$\liminf_{t\longrightarrow 0^+}\frac{\omega_{r,p}(f,w,t)}{t^{r+\varepsilon}} = 0. \tag{3.1}$$

Then f is a polynomial of degree $r-1$ a.e.

The essence of (3.1) lies in the fact that it easily follows from (1.4) that for any $P \in \Pi_{r-1}$, we have

$$\omega_{r,p}(P, w, ;) \equiv 0$$

and so (3.1) is a strong converse.

We observe that (3.1) is false for $0 < p < 1$.

Indeed set:

$$f(x) := \begin{cases} 0 & , x \in (-1, 0) \\ x^{r-1} & , x \in (0, 1). \end{cases}$$

Then $f \in L_p$, $p < 1$, and

$$\omega^r(f, t) := \sup_{0<h\leq t} \|\Delta_h^r(f)\|_{L_p(-1,1)} \;.\; t^{r-1+1/p}.$$

As f is of compact support,

$$\omega^r(f, t) \sim \omega_{r,p}(f, w, t)$$

and so (3.1) holds for any $0 < \varepsilon < -1 + 1/p$.

The essential ingredient in the proof of Theorem 1.6 is the following result which is of independent interest:

Theorem 1.7 *Let $w \in \mathcal{E}$, $1 \leq p \leq \infty$, $f \in L_{p,w}(-1, 1)$, $r \geq 1$, and $t \in (0, t_0)$. Then uniformly for $\lambda \in [1, \frac{t_0}{t}]$ and f and t*

$$\omega_{r,p}(f, w, \lambda t) \;.\; \lambda^r \left(\sup_{x\in[-1,1]} \Psi_{\lambda t,t}(x) \right)^r \omega_{r,p}(f, w, t) \tag{3.2}$$

where $\Psi_{\lambda t,t}$ was defined in (2.22).

In particular, given $\varepsilon > 0$, we have uniformly for $0 < t < t_0$, f and $\lambda \in [1, \frac{t_0}{t}]$

$$\omega_{r,p}(f, w, \lambda t) \;.\; \lambda^{r+\varepsilon} \omega_{r,p}(f, w, t). \tag{3.3}$$

Remark

We remark that the analogue of Theorem 1.7 for Erdős weights is Theorem 2.1 in [1]. Moreover in [4, Theorem 4.1.2], (3.3) is established for a large class of Freud weights with no ε on the right hand side in keeping with classical results. The reason for this unexpected extra factor is again due to the function Φ_t in the main part of the modulus which depends on t and is necessary to describe endpoint effects. These endpoint effects do not occur and are not natural for Freud weights. (3.3) then clarifies a statement of Lubinsky in [6, Section 5, pg 19] where it is claimed that (3.3) holds with $\varepsilon = 0$ and for every $\lambda > 1$.

Before we prove Theorem 1.7, we show how Theorem 1.6 follows from it.

Thus we present the:

Proof of Theorem 1.6

Our method of proof follows that of [1, Theorem 2.3] and [4, Theorem 4.2.1]. Choose $t \in (0, t_0)$ and define for $0 < p \leq \infty$

$$K_{r,p}(f, w, t^r) := \inf_{P \in \Pi_{1/t}} \left(\|(f - P)w\|_{L_p(-1,1)} + t^r \|P^{(r)} \Phi_t^r w\|_{L_p(-1,1)} \right). \quad (3.4)$$

Then it follows easily from (2.11) and [5, (5.9)] that we have uniformly for t

$$K_{r,p}(f, w, t^r) \sim \omega_{r,p}(f, w, t). \quad (3.5)$$

Now choose $t_1 \in [t, t_0]$. Then applying (3.3) with $\lambda := t_1/t$ and using (3.5) yields

$$K_{r,p}(f, w, t_1^r) = 0. \quad (3.6)$$

Now using (3.4) and (3.6), we may choose a sequence of polynomials $(P_i)_{i=1}^{\infty} \in \Pi_{1/t_1}$ such that

$$\|(f - P_i)w\|_{L_p(-1,1)} + t_1^r \|P_i^{(r)} \Phi_{t_1}^r w\|_{L_p(-1,1)} \leq 2^{-i} t_1^r. \quad (3.7)$$

Then for a.e $x \in (-1, 1)$ we have,

$$f(x) = P_i(x) + \sum_{j=i}^{\infty} (P_{j+1} - P_j)(x)$$

and so using (3.7) gives

$$\begin{aligned} \|f^{(r)} \Phi_{t_1}^r w\|_{L_p(-1,1)} \quad &. \quad \left(2^{-i} + \sum_{j=i}^{\infty} 2^{-(j+1)} + 2^{-j} \right) \\ &. \quad 2^{-i}. \end{aligned} \quad (3.8)$$

As (3.8) holds for each $i \geq 1$, we must have

$$\|f^{(r)} \Phi_{t_1}^r w\|_{L_p(-1,1)} = 0$$

which implies that for a.e $x \in (-1, 1)$

$$f^{(r)} \Phi_{t_1}^r w(x) = 0$$

or f is a polynomial of degree $r-1$ a.e $\square$.

We now present the:

The Proof of Theorem 1.7

Let $t \in (0, t_0)$, $\lambda \in [1, \frac{t_0}{t}]$, $\varepsilon > 0$ and let $n =$ the largest integer $\leq 1/t$. By (3.4) we may choose $P \in \Pi_{1/t}$ such that

$$\|(f-P)w\|_{L_p(-1,1)} + t^r \|wP^{(r)}\Phi_t^r\|_{L_p(-1,1)} \leq 2K_{r,p}(f, w, t^r). \tag{3.9}$$

Then using (2.11), (2.12) and (3.5) we may choose $R \in \Pi_{1/\lambda t}$ such that

$$\|(R-P)w\|_{L_p(-1,1)} \ . \ (\lambda t)^r \|P^{(r)} w \Phi_{\lambda t}^r\|_{L_p(-1,1)}. \tag{3.10}$$

Similarly we obtain

$$\begin{aligned} &(\lambda t)^r \|wR^{(r)}\Phi_{\lambda t}^r\|_{L_p(-1,1)} \\ &\quad . \ K_{r,p}(P, w, (\lambda t)^r) \ . \ \omega_{r,p}(P, w, \lambda t) \\ &\quad . \ (\lambda t)^r \|P^{(r)} w \Phi_{\lambda t}^r\|_{L_p(-1,1)}. \end{aligned} \tag{3.11}$$

Then using (3.10), (3.11), (2.11), (3.5) and (3.4) gives (3.2). (3.3) then follows from (3.2) and (2.23). $\square$

We present and prove the following existence theorem for derivatives.

Theorem 1.8

Let $w \in \mathcal{E}$, $0 < p \leq \infty$, $f \in L_{p,w}(-1,1)$, $n \geq n_0$ and $q = \min(1,p)$. Then if

$$\sum_{j=1}^{\infty} 2^{j(\varepsilon+kq)} n^{kq} E_{2^{j-1}n}[f]_{w,p}^q < \infty$$

for some $\varepsilon > 0$ and positive integer k the following hold:

(a)

$$f^{(k)} w \in L_p(-1,1).$$

(b)

$$\begin{aligned} &\|(f-P_n^*)^{(k)}\Phi_{\frac{1}{n}}^k w\|_{L_p(-1,1)} \\ &\quad . \left(\sum_{j=1}^{\infty} 2^{j(\varepsilon+kq)} n^{kq} E_{2^{j-1}n}[f]_{w,p}^q\right)^{\frac{1}{q}}. \end{aligned} \tag{3.12}$$

Proof

Let P_n^* be the best approximant to f satisfying (1.1). Then much as in the proof of Theorem 1.4, we write for a.e $x \in (-1,1)$,

$$f(x) = P_n^*(x) + \sum_{j=1}^{\infty} (P_{2^j n}^*(x) - P_{2^{j-1}n}^*(x)). \tag{3.13}$$

Now let $\varepsilon > 0$ and apply (3.13) together with (2.21), (2.23) and $\frac{\varepsilon}{q}$. This gives:

$$\begin{aligned} &\|(f - P_n^*)^{(k)} \Phi_{\frac{1}{n}}^k w\|_{L_p(-1,1)}^q \\ &\quad . \sum_{j=1}^{\infty} 2^{j(\varepsilon+kq)} n^{kq} \|(P_{2^j n}^* - P_{2^{j-1}n}^*) w\|_{L_p(-1,1)}^q \\ &\quad . \sum_{j=1}^{\infty} 2^{j(\varepsilon+kq)} n^{kq} E_{2^{j-1}n}^q [f]_{w,p}. \end{aligned}$$

Taking qth roots completes the proof of the theorem. $\square$

Finally we present the:

Proof of (1.14)

Let P_n^* be chosen to satisfy (1.1) so that (3.13) holds. Then using Theorem 1.7 and (1.1) we may write

$$\begin{aligned} &\|(f - P_n^*) w\|_{L_p(-1,1)} \\ &\quad . \sum_{j=1}^{\infty} \|(P_{2^j n}^* - P_{2^{j-1}n}^*) w\|_{L_p(-1,1)} \\ &\quad . \sum_{j=1}^{\infty} \omega_{r,p}(f, w, 2^{-j} n^{-1}). \end{aligned}$$

Then observing that for each fixed j

$$\int_{\frac{1}{2^j n}}^{\frac{1}{2^{j-1}n}} \frac{1}{\tau} d\tau = \log 2$$

and using Theorem 1.7, we obtain the identity

$$E_n[f]_{w,p} \; . \int_0^{1/n} \frac{\omega_{r,p}(f, w, \tau)}{\tau} d\tau. \tag{3.14}$$

Set $l := \log_2 n$. Then combining (2.11), (2.19) and (3.14) yields

$$\|P_n^{*(r)}\Phi^r_{\frac{1}{n}}w\|_{L_p(-1,1)} \tag{3.15}$$
$$. \; n^r \sum_{j=-1}^{l} (l-j+1)^{r/2} 2^{-r(2l-j)} \int_0^{1/2^j} \frac{\omega_{r,p}(f,w,\tau)}{\tau} d\tau.$$

Let $0 < \varepsilon < r/2$. Then Theorem 1.7 shows that we have for each fixed j

$$\int_0^{1/2^j} \frac{\omega_{r,p}(f,w,\tau)}{\tau} d\tau \; . \; 2^{(l-j)(r+\varepsilon)} \int_0^{1/2^l} \frac{\omega_{r,p}(f,w,\tau)}{\tau} d\tau. \tag{3.16}$$

Thus combining (3.16) and (3.15) yields (1.14). □

We close with some final comments and open problems:

As is illustrated in this paper, the modulus of smoothness (1.4) has the advantage that it illustrates endpoint effects in the Mhaskar-Rakhmanov-Saff interval by virtue of the function Φ_t. This however does introduce extra logarithmic terms in Theorem 1.4 (a) and an extra ε term in Theorem 1.6. Moreover and more importantly, it is not obvious that in $L_p(0 < p < 1)$ the modulus in (1.4) tends to zero for small t as there is a symmetric difference of f in the main part of the modulus multiplied by w. For the case $p \geq 1$ this follows by the equivalences (3.5) and, see [5, (1.24)],

$$K_{r,p}(f,w,t^r) \sim \inf_g \left(\|(f-g)w\|_{L_p(-1,1)} + t^r \|g^{(r)}\Phi_t^r w\|_{L_p(-1,1)} \right)$$

where $g^{(r-1)}$ is locally absolutely continuous.

Thus it would be interesting to investigate in detail the relationship between the modulus (1.4) and one with Φ_t replaced by Φ_h for the class $\mathcal{E}$. Moreover, in $L_p(0 < p < 1)$ it seems appropriate to replace symmetric differences in the main part of (1.4) by a backward difference operator

$$\hat{\Delta}_h^r(f,x,(-1,1)) := \begin{cases} \sum_{i=0}^r \binom{r}{i}(-1)^i f(x-ih) & , x \in (-1,1) \\ 0 & , \text{otherwise} \end{cases}$$

and then use the relation

$$w(x)|f(x-h)| \leq w(x-h)|f(x-h)|.$$

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