On the maximum modulus of weighted polynomials in the plane, a theorem of Rakhmanov, Mhaskar and Saff revisited.

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Abstract

Let $\Sigma \subseteq \mathbb{C}$ be a closed set of positive capacity at each point in Σ and $w : \Sigma \to [0, \infty)$ a continuous, weight with $|z|w(z) \to 0$, $|z| \to \infty$, $z \in \Sigma$ if Σ is unbounded. Assume further that the set where w is positive is of positive capacity. A classical theorem, obtained independently by Rakhmanov and Mhaskar and Saff says that if S_w denotes the support of the equilibrium measure for w, then $||P_n w^n||_\Sigma = ||P_n w^n||_{S_w}$ for any polynomial P_n with $\deg P_n \leq n$. This does not rule out the possibility that $|P_n w^n|$ may attain a maximum outside S_w. We prove that if in addition, Σ is regular with respect to the Dirichlet problem on $\mathbb{C}$ and if it coincides with its outer boundary, then all points where $|P_n w^n|$ attain their maxima must lie in S_w. The case when $\Sigma \subseteq \mathbb{R}$ consists of a finite union of finite or infinite intervals is due to Lorentz, von Golitschek and Makovoz. Counter examples are given to show that our requirements on Σ cannot in general be relaxed.

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1 Introduction and Statement of Main Result.

The purpose of this note, is to extend a theorem of Lorentz, von Golitschek and Makovoz, see [3, Proposition 1.4.1] dealing with the characterization of sets in the complex plane where weighted polynomials attain their maximum values. To set the scene for our investigation, let $\Sigma \subseteq \mathbb{C}$ be a closed set and $w : \Sigma \to [0, \infty)$ a continuous weight. If Σ is unbounded, assume further that $|z|w(z) \to 0$, $|z| \to \infty$, $z \in \Sigma$. We will also henceforth suppose that Σ is of positive capacity at each point in Σ, i.e., for every point $z_0 \in \Sigma$, the set $\{z \in \Sigma : |z - z_0| < \delta\}$ has positive capacity for any $\delta > 0$ and that the set where w is positive, has positive capacity. We set $Q := -\log w$ and call w *strongly* *admissible* and Q the external field associated with w. The *equilibrium measure*, see [5], in the presence of an admissible external field,

$$Q : \Sigma \longrightarrow \mathbb{R}$$

is the unique Borel probability measure μ_w with compact support on Σ satisfying for a unique constant F_w,

$$M_w(z) := U^{\mu_w}(z) + Q(z) - F_w \le 0,\ z \in S_w := \mathrm{supp}(\mu_w) \tag{1.1}$$

and

$$M_w(z) \ge 0,\ \text{q.e.}\, z \in \Sigma. \tag{1.2}$$

Here, U^{μ_w} denotes the logarithmic potential of μ_w, i.e.,

$$U^{\mu_w}(z) := \int_{\Sigma} \log \frac{1}{|z-t|} d\mu_w(t),\ z \in \mathbb{C}$$

and q.e. $z \in \Sigma$ means that (1.2) holds everywhere on Σ with the exception of a set of logarithmic capacity zero. A classical theorem, obtained independently by Rakhmanov and Mhaskar and Saff, [5, Corollary 3.2.6] is well known:

Proposition 1.1 Let w be strongly admissible. Then

$$||P_n w^n||_{\Sigma} = ||P_n w^n||_{S_w}$$

for every polynomial P_n with $\deg P_n \le n$.

Proposition 1.1 says that the sup norm of a weighted polynomial lives in the set S_w. It does not however rule out the possibility that a weighted polynomial may take a maximum outside S_w. In this note, we show that if we assume some additional structure on the underlying set Σ, namely if we assume that it is regular with respect to the Dirichlet problem on $\mathbb{C}$ and that it coincides with its outer boundary, then all points where $|P_n w^n|$ attain their maxima are contained in the set S_w. We also show by way of counter examples, that our additional assumptions on the set Σ cannot in general be removed. Throughout let Π_n denote the class of polynomials of degree at most n, $n \ge 1$.

For our main result, we need two important definitions:

(a) The outer domain Ω of Σ is the unbounded component of the complement $\overline{\mathbb{C}} \backslash \Sigma$. The *outer boundary* of Σ is defined to be $\partial\Omega$, the boundary of Ω. For example we shall need in Remark 1.3(b) below the fact (see [5, Corollary 4.5]), that if $w \equiv 1$, S_w is contained in the outer boundary of Σ.

(b) We shall say that a point $z \in \Sigma$ is regular with respect to the Dirichlet problem (or for short *regular*) on $\mathbb{C}$ if the Green's function for Σ, (see [5, pg 108]), is continuous at z. If every point in Σ is regular, then Σ is regular. For example, if Σ is simply connected or a finite union of finite or infinite real intervals, then Σ is regular.

Using the above two concepts, we shall henceforth adopt the following convention. Σ will be called *strongly regular* if it is regular and if the outer boundary of Σ coincides with Σ.

It is easy to see, in view of (a) and (b), that if Σ is simply connected with empty interior then Σ is strongly regular. Moreover if Σ is a finite union of finite or infinite intervals, then Σ is also strongly regular. Examples of sets in the plane which are strongly regular are line segments and simple closed contours. If Σ is strongly regular and w is strongly admissible, then it follows from [5, Theorems 1.4.4 and 1.5.1(iv')] that U^{μ_w} is continuous everywhere in $\mathbb{C}$ and hence that (1.2) holds everywhere on Σ.

Following is our main result:

Theorem 1.2 Let w be strongly admissible and let $m \in \mathbb{N}$.

(a) Then for every collection of polynomials $\{P_{n,k}\}_{k=1}^{m} \in \Pi_n$, $n \geq 1$

$$\left\| \sum_{k=1}^{m} |P_{n,k}| w^n \right\|_{\Sigma} = \left\| \sum_{k=1}^{m} |P_{n,k}| w^n \right\|_{S_w}. \tag{1.3}$$

(b) Assume in addition that Σ is strongly regular. Then if $x_0 \in \Sigma$ is a point where $\|\sum_{k=1}^{m} |P_{n,k}| w^n\|_{\Sigma}$ is attained, then $x_0 \in S_w$.

Remark 1.3(a)

(a) Theorem 1.2(a) for $m = 1$ is [5, Corollary 3.2.6] which was obtained independently by Rakhmanov and Mhaskar and Saff.

(b) For $m \geq 1$ and under the assumption that w is convex, positive and $\Sigma = (c, d)$ with $-\infty \leq c < 0 < d \leq \infty$, Theorem 1.2(b) follows from [2, Theorem 2.6]. When $m = 1$ and Σ is a finite union of finite or infinite real intervals, Theorem 1.2(b) has been shown earlier in [3, Proposition 4.1.1]. Our proof of Theorem 1.2(b) uses methods of logarithmic potential theory which were developed in [1, Lemma 2.2]. As is shown in Remark 1.3(b) below, it essentially cannot be improved further.

Remark 1.3(b) In this remark we explain why the strong regularity assumptions of Theorem 1.2(b) cannot be dropped in general. Indeed, let us take in Theorem 1.2(b), $w \equiv 1$. Then using [5, Corollary 4.5], we know that S_w is contained in the outer boundary of Σ. If this outer boundary was not Σ itself, one could choose P_n to be constant and then the maximum of $P_n w^n$ is attained everywhere, not just on S_w. For a particular $w \not\equiv 1$, Saff and Totik in [5, pg 157] construct an annulus with positive interior for which Theorem 1.2(b) fails.

2 Proof of Main Result

In this section, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 We will need the inequality, see [5, Theorem 3.5.1 and Corollary 3.5.3],

$$|P_n w^n(z)| \leq \exp(-nM_w(z)) \, \|P_n w^n\|_{S_w} \, , \; z \in \Sigma, \; P_n \in \Pi_n, \; n \geq 1. \tag{2.1}$$

Let $\{P_{n,k}\}_{k=1}^{m} \in \Pi_n$ be given. We will assume that $m = n$ for the general case follows in exactly the same way. Let us also choose $x_0 \in \Sigma$ for which

$$\left\| \sum_{k=1}^{n} |P_{n,k}| w^n \right\|_{\Sigma} = \sum_{k=1}^{n} |P_{n,k}|(x_0) w^n(x_0).$$

Now choose $Q_n \in \Pi_n$ such that

$$\begin{aligned} \|Q_n w^n\|_{\Sigma} = ||Q_n w^n||_{S_w} = \left\| \sum_{k=1}^{n} |P_{n,k}| w^n \right\|_{\Sigma} = \\ = \sum_{k=1}^{n} |P_{n,k}|(x_0) w^n(x_0) = |Q_n(x_0) w^n(x_0)|. \end{aligned} \tag{2.2}$$

This is done by considering

$$Q_n(x) = \sum_{k=1}^{n} \varepsilon_k P_{n,k}(x), \; \varepsilon_k = \mathrm{sign} P_{n,k}(x_0)$$

and using (2.1). See [6, Lemma 1]. Theorem 1.2(a) then follows.

The difficult task is to now show that $x_0 \in S_w$. Indeed, using (1.1), (1.2), (2.1) and (2.2) it follows that

$$x_0 \in S_w^* := \{z \in \Sigma : \; M_w(z) = 0\} \, .$$

If w is convex, positive and $\Sigma = (c, d)$ with $-\infty \leq c < 0 < d \leq \infty$ it follows from [2, Theorem 2.6] that $S_w^* = S_w$ and Theorem 1.2(b) would then follow. In general, however it is not true that $S_w^* = S_w$. We now show that indeed $x_0 \in S_w$ and in doing so we establish Theorem 1.2(b). Actually we will prove the following:

Let $\mu := \mu_w$ and suppose that

$$\begin{aligned} & U^{\mu}(x_0) + \frac{1}{n} \log |Q_n(x_0)| \\ & \geq \max\nolimits_{z \in S_w} \left(U^{\mu}(z) + \frac{1}{n} \log |Q_n(z)| \right) . \end{aligned} \tag{2.3}$$

Then $x_0 \in S_w$. To see this, consider the function

$$U^\mu + \frac{1}{n}\log|Q_n|.$$

Firstly U^μ is harmonic outside S_w and therefore $U^\mu + \frac{1}{n}\log|Q_n|$ is subharmonic outside S_w. It is also subharmonic at ∞. To see this, simply observe that

$$U^\mu + \frac{1}{n}\log|Q_n| = U^{\mu-\nu_n}$$

where ν_n is the normalized counting measure of Q_n with mass $||\nu_n|| \le ||\mu|| = 1$. See also [1, Lemma 2.2].

By the maximum principle for subharmonic functions, see [5, Theorem 1.2.4], $U^\mu + \frac{1}{n}\log|Q_n|$ attains its maximum on S_w. If the maximum is attained at a point outside S_w, then necessarily $U^\mu(z) + \frac{1}{n}\log|Q_n|(z)$ is constant for every $z \in \mathbb{C}$. If this is the case, we may let $|z| \to \infty$ and conclude that

$$U^\mu(z) = -\frac{1}{n}\log|Q_n|(z),\ \forall z \in \mathbb{C}.$$

This is clearly impossible as U^μ is continuous everywhere on $\mathbb{C}$. Thus if (2.3) holds for x_0, then $x_0 \in S_w$. Thus everything boils down to showing (2.3).

Indeed, using (1.5) and (1.6), we first see that

$$\begin{aligned}
&\max\nolimits_{z\in S_w}(U^\mu(z) + Q(z)) = F_w \\
&\le U^\mu(z) + Q(z),\ z \in \Sigma.
\end{aligned}$$

Thus applying the above and (2.2) we see that

$$\begin{aligned}
&U^\mu(x_0) + \frac{1}{n}\log|Q_n(x_0)| \\
&= \frac{1}{n}\log|Q_n(x_0)w^n(x_0)| + U^\mu(x_0) + Q(x_0) \\
&\ge \max\nolimits_{z\in S_w}\frac{1}{n}\log|Q_n(z)w^n(z)| + \max\nolimits_{z\in S_w}(U^\mu(z) + Q(z)) \\
&\ge \max\nolimits_{z\in S_w}\left[\frac{1}{n}\log|Q_n(z)w^n(z)| + U^\mu(z) + Q(z)\right] \\
&= \max\nolimits_{z\in S_w}\left[U^\mu(z) + \frac{1}{n}\log|Q_n(z)|\right].
\end{aligned}$$

Thus (2.3) holds and we have proved Theorem 1.2(b). $\square$

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