On Point Energies, Separation Radius and Mesh Norm for s-Extremal Configurations on Compact Sets in $\mathbb{R}^n$

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Abstract

We investigate bounds for point energies, separation radius, and mesh norm of certain arrangements of N points on sets A from a class $\mathcal{A}^d$ of d-dimensional compact sets embedded in $\mathbb{R}^{d'}$, $1 \le d \le d'$. We assume that these points interact through a Riesz potential $V = |\cdot|^{-s}$, where $s > 0$ and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{d'}$. With $\delta_s^*(A,N)$ and $\rho_s^*(A,N)$ denoting, respectively, the separation radius and mesh norm of *s-extremal* configurations, which are defined to yield minimal discrete Riesz s-energy, we show, in particular, the following.

(A) For the d-dimensional unit sphere $S^d \subset \mathbb{R}^{d+1}$ and $s < d-1$, $\delta_s^*(S^d,N) \ge cN^{-1/(s+1)}$ and, moreover, $\delta_s^*(S^d,N) \ge cN^{-1/(s+2)}$ if $s \le d-2$. The latter result is sharp in the case $s = d-2$. In addition, point energies for s-extremal configurations are asymptotically equal. This observation relates to numerical experiments on observed scar defects in certain biological systems.

(B) For $A \in \mathcal{A}^d$ and $s > d$, $\delta_s^*(A,N) \ge cN^{-1/d}$ and the mesh ratio $\rho_s^*(A,N)/\delta_s^*(A,N)$ is uniformly bounded for a wide subclass of $\mathcal{A}^d$. We also conclude that point energies for s-extremal configurations have the same order, as $N \to \infty$.

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1 Introduction

The problem of distributing a large number of points over the surface of a smooth manifold is an interesting and widely studied problem with numerous applications in diverse areas. To name just a few: spherical t-designs, discrepancy and combinatorics, Monte-Carlo and Quasi-Monte-Carlo methods, approximation theory, finite fields, complexity theory, frame theory, viral morphology, crystallography, molecular structure and electrostatics. We refer the reader to [1]–[11], [13]–[16], [18]–[25], and the many references cited therein for a detailed account of this fascinating subject. In this paper, we are interested in studying point energies, separation and mesh norm for arrangements of N points on a class of d-dimensional compact sets A embedded in $\mathbb{R}^{d'}$. (Here and throughout the paper, $1 \le d \le d'$ are integers.) We assume that these N-arrangements interact through the power law (Riesz) potential $V = |\cdot|^{-s}$, where $s > 0$ and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{d'}$.

Given a compact set $A \subset \mathbb{R}^{d'}$ and a collection $\boldsymbol{\omega}_N = \{x_1, \ldots, x_N\}$ of $N \ge 2$ distinct points on A, the discrete Riesz s-energy associated with $\boldsymbol{\omega}_N$ is given by

$$E_s(A, \boldsymbol{\omega}_N) := \sum_{1 \le i < j \le N} |x_i - x_j|^{-s}. \tag{1.1}$$

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Let $\boldsymbol{\omega}_s^*(A,N) := \{x_1^*, \dots, x_N^*\} \subset A$ be a configuration for which $E_s(A, \boldsymbol{\omega}_N)$ attains its minimal value, that is,

$$\mathcal{E}_s(A,N) := \min_{\boldsymbol{\omega}_N \subset A} E_s(A, \boldsymbol{\omega}_N) = E_s(A, \boldsymbol{\omega}_s^*(A,N)).$$

In accordance with convention, we shall call such minimal configurations *s-extremal configurations*. It is well-known that, in general, s-extremal configurations are not always unique. For example, in the case of the unit sphere $S^d := \{x \in \mathbb{R}^{d+1} : |x| = 1\}$, they are invariant under rotations.

In this paper, we investigate bounds for point energies, separation radius, and mesh norm of s-*extremal* configurations, which are defined to yield minimal discrete Riesz s-energy. With $\delta_s^*(A,N)$ and $\rho_s^*(A,N)$ denoting, respectively, the separation radius and mesh norm of such configurations, we show, in particular, the following. (A) For the d-dimensional unit sphere $S^d \subset \mathbb{R}^{d+1}$ and $s < d-1$, $\delta_s^*(S^d, N) \geq cN^{-1/(s+1)}$ and, moreover, $\delta_s^*(S^d, N) \geq cN^{-1/(s+2)}$ if $s \leq d-2$. The latter result is sharp in the case $s = d-2$. In addition, point energies for s-extremal configurations are asymptotically equal. This observation relates to numerical experiments on observed scar defects in certain biological systems.
(B) For $A \in \mathcal{A}^d$ and $s > d$, $\delta_s^*(A,N) \geq cN^{-1/d}$ and the mesh ratio $\rho_s^*(A,N)/\delta_s^*(A,N)$ is uniformly bounded for a wide subclass of $\mathcal{A}^d$. We also conclude that point energies for s-extremal configurations have the same order, as $N \to \infty$.

Natural questions that arise in studying the discrete Riesz energy are:

(1) What is the asymptotic behavior of $\mathcal{E}_s(A,N)$, as $N \to \infty$?

(2) How are s-extremal configurations distributed on A for large N?

It is well-known that answers to these questions essentially depend on the relation between s and the Hausdorff dimension $d_H(A)$ of A. We demonstrate this fact with the following two classical examples and refer the reader to [15] for more details.

Example 1 The interval $[-1,1]$, $d_H([-1,1]) = 1$: In the limiting cases, i.e., $s = 0$ (logarithmic interactions) and $s = \infty$ (best-packing problem), the s-extremal configurations are Fekete points and equally spaced points, respectively. It is well-known that Fekete points are distributed on $[-1,1]$ according to the arcsine measure, which has the density $\mu_0'(x) := (1/\pi)(1-x^2)^{-1/2}$, while the equally spaced points, $-1 + 2(k-1)/(N-1)$, $k = 1, \dots, N$, have the arclength distribution, as $N \to \infty$. It is also known that $s = 1$ is the critical value in the sense that s-extremal configurations are distributed on $[-1,1]$ differently for $s < 1$ and $s \geq 1$ (see [17, Appendix] and [20]). Indeed, for $s < 1$, the limiting distribution of s-extremal configurations has an arcsine-type density

$$\mu_s'(x) := \frac{\boldsymbol{\Gamma}(1+s/2)}{\sqrt{\pi}\,\boldsymbol{\Gamma}((1+s)/2)}\,(1-x^2)^{(s-1)/2} \tag{1.2}$$

and, for $s \geq 1$, the limiting distribution is the arclength distribution.

Concerning the minimal energies, they again behave differently for $s < 1$, $s = 1$, and $s > 1$:

$$\mathcal{E}_s([-1,1],N) \asymp \begin{cases} (1/2)N^2 e_s\,, & s < 1, \\ (1/2)N^2 \ln N, & s = 1, \\ (1/2)^s \zeta(s) N^{1+s}, & s > 1, \end{cases}$$

where $e_s := [\sqrt{\pi}\Gamma(1+s/2)]\,/\,[\cos(\pi s/2)\Gamma((1+s)/2)]$ and $\zeta(s)$ stands for the Riemann zeta function.

This dependence of the distribution of s-extremal configurations over $[-1,1]$ and the asymptotics for minimal discrete s-energy on s can be easily explained from potential theory point of view. Indeed, for a probability Borel measure ν on $[-1,1]$, its s-energy integral is defined to be

$$I_s([-1,1],\nu) := \iint\limits_{[-1,1]^2} |x-y|^{-s} d\nu(x) d\nu(y) \tag{1.3}$$

(which can be finite or infinite). Let, for a set of points $\boldsymbol{\omega}_N = \{x_1, \dots, x_N\}$ on $[-1,1]$,

$$\nu^{\boldsymbol{\omega}_N} := \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\delta}_{x_i}$$

denote the normalized counting measure of $\boldsymbol{\omega}_N$ (so that $\nu^{\boldsymbol{\omega}_N}([-1,1]) = 1$). Then the discrete Riesz s-energy, associated with $\boldsymbol{\omega}_N$ and defined by (1.1), can be written as

$$E_s([-1,1], \boldsymbol{\omega}_N) = (1/2)N^2 \iint_{x\neq y} |x-y|^{-s} d\nu^{\boldsymbol{\omega}_N}(x) d\nu^{\boldsymbol{\omega}_N}(y), \tag{1.4}$$

where the integral represents a discrete analog of the s-energy integral (1.3) for the point-mass measure $\nu^{\boldsymbol{\omega}_N}$.

If $s < 1$, then it is well-known (cf. [17, Appendix]) that the energy integral (1.3) is minimized uniquely by an arcsine-type measure ν_s^*, whose density $\mu_s'(x)$ with respect to the Lebesgue measure is given by (1.2). On the other hand, the normalized counting measure $\nu_{s,N}^*$ of an s-extreme configuration minimizes the discrete energy integral in (1.4) over all configurations $\boldsymbol{\omega}_N$ on $[-1,1]$. Thus one can reasonably expect that, for N large, $\nu_{s,N}^*$ is "close" to ν_s^* and, therefore, the minimal discrete s-energy $\mathcal{E}_s([-1,1], N)$ is close to $(1/2)N^2 I_s([-1,1], \nu_s^*) = (1/2)N^2 e_s$.

If $s \geq 1$, then the energy integral (1.3) diverges for every measure ν. Thus, $\mathcal{E}_s([-1,1], N)$ must grow faster than N^2. Concerning the distribution of s-extremal points over $[-1,1]$, the interactions are now strong enough to force them to stay away from each other as far as possible. Of course, depending on s, "far" neighbors still incorporate some energy in $\mathcal{E}_s([-1,1], N)$, but the closest neighbors are dominating. So, s-extremal points distribute themselves over $[-1,1]$ in an equally spaced manner.

Example 2 The unit sphere S^d, $d_H(S^d) = d$: Here we again see three distinct cases: $s < d$, $s = d$, and $s > d$. Although it turns out that, for any s, the limiting distribution of s-extremal configurations is given by the normalized area measure on S^d (cf. [17]; [13], [9]; [14], resp.), which is not a big surprise due to the rotation invariance, the asymptotic behavior of $\mathcal{E}_s(S^d, N)$ is quite different. With

$$\tau_{s,d}(N) := \begin{cases} N^2, & s < d, \\ N^2 \ln N, & s = d, \\ N^{1+s/d}, & s > d, \end{cases}$$

it is known that the limit $\lim_{N\to\infty} \mathcal{E}_s(S^d, N)/\tau_{s,d}(N)$ exists (see [17], [16], [14]). Moreover, in the first two cases, it has the value $(1/2)\gamma_{s,d}$, where

$$\begin{aligned} \gamma_{d,d} &= \frac{\Gamma((d+1)/2)}{d\sqrt{\pi}\,\Gamma(d/2)}, \\ \gamma_{s,d} &= \frac{\Gamma((d+1)/2)\Gamma(d-s)}{\Gamma((d-s+1)/2)\Gamma(d-s/2)}, \quad s < d. \end{aligned} \tag{1.5}$$

The dependence of the growth rate of $\mathcal{E}_s(S^d, N)$ on s can be explained using potential theory arguments similar to those in Example 1.

2 Class $\mathcal{A}^d$

In this section, we introduce a class of compact sets $A \subset \mathbb{R}^{d'}$, for which, in the case $s \geq d$ and under some additional assumptions, the asymptotic behavior of $\mathcal{E}_s(A, N)$, separation results, and the limiting distribution of $\boldsymbol{\omega}_s^*(A, N)$ over A (in terms of weak-star convergence of the normalized counting measures) have been recently obtained (see [14, Theorems 2.1–2.4]). We will also give two important properties of sets in $\mathcal{A}^d$ (estimates (2.3) and (2.4) below), which turn out to be crucial in what follows.

For a set $A \subset \mathbb{R}^{d'}$, let $\mathcal{H}^d(A)$ denote its d-dimensional Hausdorff measure (which reduces to d-dimensional Lebesgue measure if $d' = d$).

Definition 2.1 *We say that a set A belongs to the class $\mathcal{A}^d$ if, for some $d' \geq d$, $A \subset \mathbb{R}^{d'}$ and*

(1) $\mathcal{H}^d(A) > 0$ *and*

(2) *A is a finite union of bi-Lipschitz images of compact sets in $\mathbb{R}^d$, that is*

$$A = \bigcup_{i=1}^{m} \boldsymbol{\varphi}_i\left(K_i\right),$$

where each $K_i \subset \mathbb{R}^d$ is compact and $\boldsymbol{\varphi}_i : K_i \to \mathbb{R}^{d'}$ is bi-Lipschitz on K_i, $i = 1, \ldots, m$.

We recall that a mapping $\boldsymbol{\varphi} : K \to \mathbb{R}^{d'}$ is called bi-Lipschitz (with a constant L) on a compact set $K \subset \mathbb{R}^d$ if there exists a constant L such that for all x, $y \in K$

$$(1/L)|x-y| \leq |\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(y)| \leq L|x-y|.$$

Clearly, $\mathcal{H}^d(\boldsymbol{\varphi}(K')) \leq L^d \mathcal{H}^d(K')$ for any measurable set $K' \subseteq K$. In particular, it follows that, for any measurable set $A' \subseteq \boldsymbol{\varphi}(K)$,

$$\mathcal{H}^d(A') \leq C\left[\operatorname{diam}(A')\right]^d, \tag{2.1}$$

where the constant C depends on L and d only, and $\operatorname{diam}(\cdot)$ means the diameter of a set. Indeed, since $\operatorname{diam}\left(\boldsymbol{\varphi}^{-1}(A')\right) \leq L\operatorname{diam}(A')$, the preimage $\boldsymbol{\varphi}^{-1}(A')$ is contained in a ball $B \subset \mathbb{R}^d$ of radius $L\operatorname{diam}(A')$. Thus,

$$\mathcal{H}^d(A') \leq L^d \mathcal{H}^d\left[\boldsymbol{\varphi}^{-1}(A')\right] \leq L^d \mathcal{H}^d(B) \leq \frac{2\pi^{d/2} L^{2d}}{d\Gamma(d/2)}\left[\operatorname{diam}(A')\right]^d.$$

We now give some examples of sets from the class $\mathcal{A}^d$.

Example 3 (i) Compact sets A in $\mathbb{R}^d$ with $\mathcal{H}^d(A) > 0$: with $d' = d$, these sets are bi-Lipschitz images of themselves under the identity map. (One can also consider these sets embedded in $\mathbb{R}^{d'}$ for $d' > d$.) For example, balls $B^d(x,r) := \{y \in \mathbb{R}^d : |y-x| \leq r\}$, d-dimensional cubes and parallelepipeds, d-dimensional Cantor sets having positive $\mathcal{H}^d$-measure.

(ii) d-dimensional spheres in $\mathbb{R}^{d+1}$ (more generally, ellipsoids), since a closed hemisphere is a bi-Lipschitz image of a d-dimensional ball under a stereographic projection.

(iii) Quasismooth (chord-arc) curves in $\mathbb{R}^{d'}$. These are Jordan curves $A \subset \mathbb{R}^{d'}$ satisfying the following condition: there exists a constant C such that, for any two points x, $y \in A$, the length of the (shortest) subarc of A with endpoints x and y is bounded by $C|x-y|$. In this case, the bi-Lipschitz mapping $\boldsymbol{\varphi} : [0, \text{length(A)}] \to \mathbb{R}^{d'}$ is given by a natural parametrization of the curve.

(iv) Finite unions of sets from (i)–(iii). For instance, a ball $B^d(x,r)$ with a quasismooth outgoing "tail". Thus, sets in $\mathcal{A}^d$ can have, locally, different Hausdorff dimensions.

We remark the following useful properties of sets $A \in \mathcal{A}^d$. For $r > 0$ and $x \in A$, let

$$E(x,r) := \{y \in A : |y-x| < r\} \tag{2.2}$$

denote a "cap" on A with center x and radius r. Then

$$\mathcal{H}^d\left(E(x,r)\right) \leq C_1 r^d \tag{2.3}$$

with a constant C_1 independent of x and r. Indeed, let L be such that all $\boldsymbol{\varphi}_i$'s in Definition 2.1 are bi-Lipschitz with the constant L. We note that $\operatorname{diam}(E(x,r)) \leq 2r$ and, using (2.1), conclude that

$$\mathcal{H}^d\left(E(x,r)\right) \leq \sum_{i=1}^{m} \mathcal{H}^d\left(E(x,r) \cap K_i\right) \leq mC(2r)^d = C_1 r^d.$$

In particular, $\mathcal{H}^d(A) \le C_1[\mathrm{diam}(A)]^d < \infty$ since A is compact. Moreover, (2.3) implies that, for $s > d$,

$$\int\limits_{A\setminus E(x,r)} |y-x|^{-s} d\mathcal{H}^d(y) \le C_2 r^{d-s}, \tag{2.4}$$

where C_2 does not depend on x and r. This inequality can be verified as follows.

$$\begin{aligned}\int\limits_{A\setminus E(x,r)} |y-x|^{-s} d\mathcal{H}^d(y) &= \int\limits_0^\infty \mathcal{H}^d\left(\left\{|y-x|^{-s} > t\right\} \cap \{A \setminus E(x,r)\}\right) dt \\ &\le \int\limits_0^{r^{-s}} \mathcal{H}^d\left(E(x, t^{-1/s})\right) dt \le \int\limits_0^{r^{-s}} C_1 t^{-d/s} dt = C_2 r^{d-s}.\end{aligned}$$

3 Main results

Our results concern point energies, separation radius, and the mesh norm for sets $A \in \mathcal{A}^d$ in the case $s > d$. We also obtain new separation estimates for the unit sphere $S^d \subset \mathbb{R}^{d+1}$ in the case $s < d-1$ and show that point energies are asymptotically equal, as $N \to \infty$.

Throughout the paper, we denote by C, C_1, ... positive constants, and by c, c_1, ... sufficiently small positive constants (different each time, in general), that may depend on d, s, A, and other parameters not essential for arguments, but independent of N and other variable quantities.

We define the *point energies* associated with $\boldsymbol{\omega}_s^*(A,N)$ by

$$\mathcal{E}_{j,s}(A,N) := \sum_{\substack{i=1 \\ i\ne j}}^{N} \left|x_j^* - x_i^*\right|^{-s}, \qquad j = 1, \ldots, N. \tag{3.1}$$

For $s > d$, it was shown in [14] that, if $A \in \mathcal{A}^d$ and $m = 1$ in Definition 2.1 (i.e., $A \subset \mathbb{R}^{d'}$ is a bi-Lipschitz image of one compact set in $\mathbb{R}^d$), then

$$\mathcal{E}_{j,s}(A,N) \le CN^{s/d}, \qquad 1 \le j \le N.$$

We extend this result with

Theorem 3.2 *Let $A \in \mathcal{A}^d$ and $s > d$. Then, for all $1 \le j \le N$,*

$$\mathcal{E}_{j,s}(A,N) \le CN^{s/d}. \tag{3.2}$$

Remark 3.3 Our proof of Theorem 3.2 shows that given d, s, and A in advance, the constant C in (3.2) can be explicitly estimated. Since $\mathcal{E}_s(A,N) = (1/2)\sum_{j=1}^N \mathcal{E}_{j,s}(A,N)$, we conclude that

$$\limsup_{n\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} \le C(d,s,A)$$

with an explicit value of the constant $C(d,s,A)$, and so we obtain an estimate for a constant in the upper bound for the minimal energies.

For $j = 1, \ldots, N$ and a set $\boldsymbol{\omega}_N = \{x_1, \ldots, x_N\}$ of distinct points on $A \in \mathcal{A}^d$, we let

$$\delta_j(\boldsymbol{\omega}_N) := \min_{i \neq j} \{|x_i - x_j|\}$$

and define

$$\delta(\boldsymbol{\omega}_N) := \min_{1 \le j \le N} \delta_j(\boldsymbol{\omega}_N).$$

The quantity $\delta(\boldsymbol{\omega}_N)$ is called the *separation radius* and gives the minimal distance between points in $\boldsymbol{\omega}_N$.

We also define the *mesh norm* $\rho(A, \boldsymbol{\omega}_N)$ of $\boldsymbol{\omega}_N$ by

$$\rho(A, \boldsymbol{\omega}_N) := \max_{y \in A} \min_{x \in \boldsymbol{\omega}_N} |y - x|. \tag{3.3}$$

Geometrically, $\rho(A, \boldsymbol{\omega}_N)$ means the maximal radius of a cap $E(y, r)$ (see (2.2)) on A, which does not contain points from $\boldsymbol{\omega}_N$.

These two quantities, $\delta(\boldsymbol{\omega}_N)$ and $\rho(A, \boldsymbol{\omega}_N)$, give a good enough description of the distribution of $\boldsymbol{\omega}_N$ over the set A. It is worth mentioning that, even for a sequence $\{\boldsymbol{\omega}_N\}$ of *asymptotically* s-extremal configurations, i.e., configurations satisfying

$$\lim_{N \to \infty} \frac{E_s(A, \boldsymbol{\omega}_N)}{\mathcal{E}_s(A, N)} = 1,$$

using results from [14], one can get only trivial estimates for the separation radius. Namely,

$$\delta(\boldsymbol{\omega}_N) \ge c N^{-(1/d + 1/s)}, \qquad s > d.$$

However, for s-extremal configurations on $A \in \mathcal{A}^d$ much better (best possible) estimate for the separation radius holds. As an immediate consequence of Theorem 3.2, we get

Corollary 3.4 *For $A \in \mathcal{A}^d$, $s > d$, and any s-extremal configuration $\boldsymbol{\omega}_s^*(A, N)$ on A,*

$$\delta_s^*(A, N) := \delta(\boldsymbol{\omega}_s^*(A, N)) \ge c N^{-1/d}. \tag{3.4}$$

Indeed, for any $1 \le j \le N$, (3.2) yields

$$C N^{s/d} \ge \mathcal{E}_{j,s}(A, N) \ge [\delta_j(\boldsymbol{\omega}_s^*(A, N))]^{-s},$$

and so $\delta_j(\boldsymbol{\omega}_s^*(A, N)) \ge c N^{-1/d}$ for all j.

Remark 3.5 We remark that (3.4) was proved in [16, Corollary 1.4] for the unit sphere S^d. The same estimate was also shown in [14, Theorem 2.4] for $A \in \mathcal{A}^d$ with an additional condition that $m = 1$ in Definition 2.1. Unfortunately, this latter condition is not satisfied for many sets, including the sphere S^d, and so Corollary 3.4 constitutes an extension of [14, Theorem 2.4].

Separation results for $s < d$ are far more difficult to find in the literature. It trivially follows from Example 2 that

$$\delta(\boldsymbol{\omega}_N) \ge c N^{-2/s}$$

for any sequence $\{\boldsymbol{\omega}_N\}$ of asymptotically s-extremal configurations on the unit sphere S^d. But, even for s-extremal configurations on S^d, there are only two non-trivial results known to the authors: for $d = 2$, $s = 0$ ([10], [22]) and $d \ge 2$, $s = d - 1$ ([8]) it was shown that

$$\delta_s^*(S^d, N) \ge c N^{-1/d}. \tag{3.5}$$

(In [8], two-sided estimates were obtained for more general surfaces.) The estimate (3.5) is quite natural and, intuitively, should be valid for any $s > 0$. A reason for such a lack of results for weak interactions ($s < d$) is that this case require more delicate considerations based on the minimizing property of $\boldsymbol{\omega}_s^*(A, N)$ while strong interactions ($s > d$) prevent points to be very close to each other without affecting the total energy, and separation estimates can be obtained by looking at nearest neighbors only.

Our next result provides a separation estimate in the case $s < d - 1$ for the unit sphere S^d.

Theorem 3.6 *For $d \geq 2$ and $s < d-1$,*

$$\delta_s^*(S^d, N) \geq cN^{-1/(s+1)}. \tag{3.6}$$

As a by-product of the proof of Theorem 3.6, we obtain the following

Corollary 3.7 *For any $0 < s < d-1$,*

$$\lim_{N\to\infty} \frac{\max_{1\leq j\leq N} \mathcal{E}_{j,s}(S^d, N)}{\min_{1\leq j\leq N} \mathcal{E}_{j,s}(S^d, N)} = 1. \tag{3.7}$$

Indeed, it follows from (4.3), (4.6), and (4.11) that, for N large enough,

$$1 \leq \frac{\max_{1\leq j\leq N} \mathcal{E}_{j,s}(S^d, N)}{\min_{1\leq j\leq N} \mathcal{E}_{j,s}(S^d, N)} = \frac{\max_{1\leq j\leq N} U_j(x_j^*)}{\min_{1\leq j\leq N} U_j(x_j^*)} \leq 1 + CN^{-1/(s+1)}.$$

Remark 3.8 Numerical computations for a sphere and a torus (see [3], [5], [6], [15]) suggest that, for any $s > 0$, the point energies are nearly equal for almost all points (which are of so called "hexagonal" type). However, some points ("pentagonal") have elevated energies and some ("heptagonal") have low energies. The transition from points that are "hexagonal" to those that are "pentagonal" and "heptagonal" induces scar defects, which are conjectured to vanish for N large enough. Thus, Corollary 3.7 confirms this conjecture for $0 < s < d-1$.

The estimate (3.6) can be improved for $d \geq 3$ and $s \leq d-2$.

Theorem 3.9 *Let $d \geq 3$ and $s \leq d-2$. Then*

$$\delta_s^*(S^d, N) \geq cN^{-1/(s+2)}. \tag{3.8}$$

Note that, when $s = d-2$, (3.8) gives the best possible estimate (3.5).

Concerning the mesh norm $\rho(A, \boldsymbol{\omega}_N)$, for any sequence $\{\boldsymbol{\omega}_N\}$ of asymptotically s-extremal configurations on A, clearly,

$$\lim_{N\to\infty} \rho(A, \boldsymbol{\omega}_N) = 0.$$

However, no estimate on the rate of convergence can be made. We show that, for $s > d$, under an additional condition on $A \in \mathcal{A}^d$, the mesh norm and the separation radius of any sequence $\{\boldsymbol{\omega}_s^*(A, N)\}$ of s-extremal configurations on A have the same order, as $N \to \infty$.

Theorem 3.10 *Let $s > d$, $A \in \mathcal{A}^d$, and suppose further that*

$$\mathcal{H}^d(E(x, r)) \geq cr^d, \tag{3.9}$$

where a constant $c > 0$ is independent of $x \in A$ and $r > 0$ small enough. Then, for any s-extremal configuration $\boldsymbol{\omega}_s^*(A, N)$ on A,

$$\rho_s^*(A, N) := \rho(A, \boldsymbol{\omega}_s^*(A, N)) \leq CN^{-1/d}. \tag{3.10}$$

We remark that the condition (3.9) implies the existence of a constant $C_1 > 0$, which depends only on C in (2.1) and c in (3.9), such that

$$\max_{1\le j\le N} \delta_j\left(\boldsymbol{\omega}_s^*(A,N)\right) \le C_1 \rho_s^*(A,N), \tag{3.11}$$

Indeed, let k be an index for which the maximum in (3.11) is attained. Thus,

$$\max_{1\le j\le N} \delta_j\left(\boldsymbol{\omega}_s^*(A,N)\right) = \delta_k\left(\boldsymbol{\omega}_s^*(A,N)\right) =: \delta^*.$$

The cap $E(x_k,(1/3)\delta^*)$ contains a point y satisfying

$$|y-x_k| \ge \frac{1}{8}\left(\frac{c}{C}\right)^{1/d}\delta^*, \tag{3.12}$$

where constants $C \ge 1$ and $c \le 1$ are such that (2.1) and (3.9) hold. (Assuming that no such point existed, we would easily conclude that $\operatorname{diam}(E(x_k,(1/3)\delta^*) \le (c/C)^{1/d}\delta^*/4$ and so, by (2.1),

$$\mathcal{H}^d\left(E(x_k,(1/3)\delta^*)\right) \le c\left(\frac{\delta^*}{4}\right)^d,$$

which contradicts (3.9).) Since $|y-x_i^*| \ge (2/3)\delta^*$ for all $i \ne k$, taking into account (3.12), we obtain

$$\min_{1\le i\le N} |y-x_i^*| \ge \frac{1}{8}\left(\frac{c}{C}\right)^{1/d}\delta^*,$$

and (3.11) follows.

Corollary 3.4, Theorem 3.10, and (3.11) yield

Corollary 3.11 *Let $s > d$, and assume that $A \in \mathcal{A}^d$ satisfies* (3.9)*. Then, for any s-extremal configuration $\boldsymbol{\omega}_s^*(A,N)$ on A,*

$$c \le N^{1/d}\delta_j\left(\boldsymbol{\omega}_s^*(A,N)\right) \le C, \qquad 1 \le j \le N. \tag{3.13}$$

Since the upper estimate in (3.13) gives

$$\mathcal{E}_{j,s}(A,N) \ge \left[\delta_j\left(\boldsymbol{\omega}_s^*(A,N)\right)\right]^{-s} \ge \left[CN^{-1/d}\right]^{-s} = cN^{s/d}, \qquad 1 \le j \le N,$$

combining this inequality with (3.2), we get

Corollary 3.12 *For $s > d$, $A \in \mathcal{A}^d$ satisfying* (3.9)*, and any s-extremal configuration $\boldsymbol{\omega}_s^*(A,N)$ on A, there holds*

$$1 \le \frac{\max_{1\le j\le N}\mathcal{E}_{j,s}(A,N)}{\min_{1\le j\le N}\mathcal{E}_{j,s}(A,N)} \le C. \tag{3.14}$$

Remark 3.13 Corollary 3.12 says that, for $s > d$ and a set $A \in \mathcal{A}^d$ satisfying (3.9), the point energies are asymptotically of the same order, as $N \to \infty$. Most likely, this is the best possible assertion in the sense that an analog of (3.7) does not hold, in general.

Remark 3.14 Simple examples show that the upper estimates in (3.13) and (3.14) are not valid, in general, for a set $A \in \mathcal{A}^d$ without an additional condition on its geometry. For instance, if $x \in \mathbb{R}^{d+1}$ and $|x| > 1$, then $A := S^d \cup \{x\}$, clearly, does not enjoy these properties.

The measure condition (3.9) in Theorem 3.10 can be omitted if $m = 1$ in Definition 2.1 (in particular, if $A \subset \mathbb{R}^d$ is a compact set with positive d-dimensional Lebesgue measure).

Theorem 3.15 *Let $A \in \mathcal{A}^d$ and suppose that A is the image of one compact set $K \subset \mathbb{R}^d$ under a bi-Lipschitz mapping $\boldsymbol{\varphi}$. Then* (3.10) *holds.*

The remainder of this paper is devoted to proofs of our results.

4 Proofs

First, we introduce some notations and properties of s-extreme configurations, which will be used in subsequent proofs.

Let $A \in \mathcal{A}^d$, $\boldsymbol{\omega}_s^*(A,N) = \{x_1^*, \ldots, x_N^*\}$ be an s-extremal configuration on A,

$$\nu_{s,N}^* := \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\delta}_{x_i^*}$$

denote the normalized counting measure of $\boldsymbol{\omega}_s^*(A,N)$, and let

$$U^{\nu_{s,N}^*}(x) := \int\limits_{\mathbb{R}^{d'}} |x-y|^{-s} d\nu_{s,N}^*(y) = \int\limits_{A} |x-y|^{-s} d\nu_{s,N}^*(y) = \frac{1}{N}\sum_{i=1}^{N} |x-x_i^*|^{-s} \tag{4.1}$$

be the potential associated with $\nu_{s,N}^*$. We also introduce functions

$$U_j(x) := U^{\nu_{s,N}^*}(x) - \frac{1}{N}\left|x-x_j^*\right|^{-s} = \frac{1}{N}\sum_{\substack{i=1\\ i\neq j}}^{N} |x-x_i^*|^{-s} = \int\limits_{A} |x-y|^{-s} d\nu_{s,j,N}^*(y), \tag{4.2}$$

where $\nu_{s,j,N}^* := \nu_{s,N}^* - (1/N)\boldsymbol{\delta}_{x_j^*}$. Note that the total mass of $\nu_{s,j,N}^*$ is $(N-1)/N$. It follows from (3.1) that

$$\mathcal{E}_{j,s}(A,N) = NU_j\left(x_j^*\right). \tag{4.3}$$

Since $\boldsymbol{\omega}_s^*(A,N)$ is an s-extremal configuration, $U_j(x)$ attains its minimal value at x_j^*. Thus, for any $x \in A$ and $1 \le j \le N$, we have

$$U^{\nu_{s,N}^*}(x) = U_j(x) + \frac{1}{N}\left|x-x_j^*\right|^{-s} \ge U_j\left(x_j^*\right) + \frac{1}{N}\left|x-x_j^*\right|^{-s} = \frac{1}{N}\left[\mathcal{E}_{j,s}(A,N) + \left|x-x_j^*\right|^{-s}\right]. \tag{4.4}$$

Therefore,

$$\sum_{j=1}^{N} U^{\nu_{s,N}^*}(x) \ge \frac{1}{N}\sum_{j=1}^{N} \mathcal{E}_{j,s}(A,N) + \frac{1}{N}\sum_{j=1}^{N}\left|x-x_j^*\right|^{-s} = \frac{2}{N}\mathcal{E}_s(A,N) + U^{\nu_{s,N}^*}(x)$$

and so

$$U^{\nu_{s,N}^*}(x) \ge \frac{2}{N(N-1)}\,\mathcal{E}_s(A,N) > \frac{2}{N^2}\,\mathcal{E}_s(A,N), \qquad x \in A. \tag{4.5}$$

4.1 Proof of Theorem 3.2

Our proof will follow from (2.3), (2.4), and ideas used in [16, Section 5]. For the reader's convenience, we present a sketch of the proof.

Let $A \in \mathcal{A}^d$, $\boldsymbol{\omega}_s^*(A,N) = \{x_1^*, \ldots, x_N^*\}$ be an s-extremal configuration on A, and let $U_j(x)$ be defined by (4.2), $j = 1, \ldots, N$. We denote $D_i := A \setminus E\left(x_i^*, [2NC_1/\mathcal{H}^d(A)]^{-1/d}\right)$, where C_1 is the constant from (2.3), and $D := \cap_{i=1}^{N} D_i$. Then, using (2.4), we get

$$\begin{aligned}\int\limits_{D} U_j(x)d\mathcal{H}^d(x) &= \frac{1}{N}\sum_{\substack{i=1\\ i\neq j}}^{N} \int\limits_{D} |x-x_i^*|^{-s}\, d\mathcal{H}^d(x) \le \frac{1}{N}\sum_{\substack{i=1\\ i\neq j}}^{N} \int\limits_{D_i} |x-x_i^*|^{-s}\, d\mathcal{H}^d(x) \\ &\le \frac{1}{N}\sum_{\substack{i=1\\ i\neq j}}^{N} C_2\left[\frac{2NC_1}{\mathcal{H}^d(A)}\right]^{(s-d)/d} < C_2\left[\frac{2C_1}{\mathcal{H}^d(A)}\right]^{(s-d)/d} N^{s/d-1}.\end{aligned}$$

On the other hand, by (2.3),

$$\mathcal{H}^d(D) = \mathcal{H}^d(A) - \mathcal{H}^d\left(\bigcup_{i=1}^{N} E\left(x_i^*, [2NC_1/\mathcal{H}^d(A)]^{-1/d}\right)\right) \geq \mathcal{H}^d(A) - \sum_{i=1}^{N} C_1[2NC_1/\mathcal{H}^d(A)]^{-1} = \frac{\mathcal{H}^d(A)}{2},$$

and so the minimizing property of $\boldsymbol{\omega}_s^*(A,N)$ yields

$$U_j(x_j^*) \leq \frac{1}{\mathcal{H}^d(D)} \int_D U_j(x) d\mathcal{H}^d(x) \leq (C_1C_2)^{(s-d)/d} \left[\mathcal{H}^d(A)/2\right]^{-s/d} N^{s/d-1} = C_3 N^{s/d-1}.$$

(Note that, since C_1 and C_2 depend explicitly on d, s, and A, so does C_3.)

Finally, using (4.3), we obtain (3.2). ■

4.2 Proof of Theorems 3.6 and 3.9

Note that, since $d' = d+1$ and $s < d-1 = d'-2$, the potential

$$U^\nu(x) := \int_{\mathbb{R}^{d+1}} |x-y|^{-s} d\nu(y)$$

is superharmonic in $\mathbb{R}^{d+1}$ for any finite positive Borel measure ν on $\mathbb{R}^{d+1}$ (see [17, Thm. 1.4]).

To save in writing, we will use a standard notation σ for the normalized d-dimensional Hausdorff measure $\mathcal{H}^d$ on S^d.

Let $\gamma_{s,d}$ be defined by (1.5). It is well-known that (for any $0 < s < d$) $\sigma(x)$ is the equilibrium measure for S^d, and so $U^\sigma(x) \equiv \gamma_{s,d}$ on S^d. Using this fact and integrating both sides of

$$\frac{1}{N}\sum_{i=1}^{N} |x - x_i^*|^{-s} = U^{\nu_{s,N}^*}(x) \geq \frac{1}{N}\left[\mathcal{E}_{j,s}(A,N) + \left|x - x_j^*\right|^{-s}\right] = U_j(x_j^*) + \frac{1}{N}\left|x - x_j^*\right|^{-s}$$

(see (4.1), (4.4), and (4.3)) against $\sigma(x)$, we get

$$\gamma_{s,d} \geq U_j(x_j^*) + \frac{1}{N}\gamma_{s,d}.$$

Thus,

$$U_j\left(x_j^*\right) \leq \frac{N-1}{N}\gamma_{s,d} < \gamma_{s,d}, \qquad j = 1,\ldots,N. \tag{4.6}$$

Next, we obtain estimates for $U_j\left(x_j^*\right)$ from below. It was shown in ([16], [25]) that

$$\frac{1}{2}\gamma_{s,d}N^2 - C_1 N^{1+\alpha(s,d)} \leq \mathcal{E}_s\left(S^d, N\right) \leq \frac{1}{2}\gamma_{s,d}N^2 - c_1 N^{1+s/d}, \tag{4.7}$$

$$\alpha(s,d) = \begin{cases} s/(s+2), & 0 < s < d-2, \\ s/d, & d-2 \leq s < d \end{cases}$$

For $d-2 \leq s < d$, (4.7) gives the exact bounds for the second term in the asymptotic behavior of $\mathcal{E}_s(S^d,N)$. In the case $0 < s < d-2$, the lower estimate in (4.7), most likely, is not best possible.

We now use estimates (4.5) and (4.7) to conclude that, for $x \in S^d$,

$$U^{\nu_{s,N}^*}(x) > \frac{2}{N^2}\left(\frac{1}{2}\gamma_{s,d}N^2 - C_1N^{1+\alpha(s,d)}\right) = \gamma_{s,d} - C_2 N^{\alpha(s,d)-1}. \tag{4.8}$$

Since $U^{\nu^*_{s,N}}(x)$ is superharmonic, by the minimal value principle (4.8) also holds for $|x| \le 1$.

Next, fixed $\varepsilon_0 > 0$ small enough, let $0 < \varepsilon < \varepsilon_0$. For any x satisfying $1 < |x| \le 1 + \varepsilon$, we denote $\widetilde{x} := (2 - |x|)x/|x|$. It can be easily verified that

$$\min_{y \in S^d} \frac{|\widetilde{x} - y|}{|x - y|} = \frac{3 - |x|}{|x| + 1} = 1 - \frac{2(|x| - 1)}{|x| + 1} > 1 - \varepsilon$$

(with the minimum attained at $y = -x/|x|$). Therefore, using (4.8), we get

$$\begin{aligned} U^{\nu^*_{s,N}}(x) &= N^{-1} \sum_{i=1}^{N} |x - x_i^*|^{-s} = N^{-1} \sum_{i=1}^{N} |\widetilde{x} - x_i^*|^{-s} \left(\frac{|\widetilde{x} - x_i^*|}{|x - x_i^*|} \right)^s \\ &\ge (1-\varepsilon)^s U^{\nu^*_{s,N}}(\widetilde{x}) \ge (1 - C_s \varepsilon) \left(\gamma_{s,d} - C_2 N^{\alpha(s,d)-1} \right) \ge \gamma_{s,d} - C_2 N^{\alpha(s,d)-1} - C_3 \varepsilon. \end{aligned}$$

Thus,

$$U^{\nu^*_{s,N}}(x) \ge \gamma_{s,d} - C_2 N^{\alpha(s,d)-1} - C_3 \varepsilon, \qquad |x| \le 1 + \varepsilon. \tag{4.9}$$

We now apply this estimate to establish Theorems 3.6 and 3.9. Let us assume that, for some $\delta > 0$, i and j, $|x_i^* - x_j^*| \le (1/2)\delta$, and consider the function

$$U_{i,j}(x) := U_j(x) - N^{-1} |x - x_i^*|^{-s} = U^{\nu^*_{s,N}}(x) - N^{-1} \left(\left| x - x_j^* \right|^{-s} + |x - x_i^*|^{-s} \right) = N^{-1} \sum_{k \ne i,j} |x - x_k^*|^{-s}.$$

Then it follows from (4.9) that, for any x satisfying $|x - x_j^*| = \delta$ and $|x| \le 1 + \varepsilon$,

$$\begin{aligned} U_{i,j}(x) &\ge \gamma_{s,d} - C_2 N^{\alpha(s,d)-1} - C_3 \varepsilon - N^{-1} \left[\delta^{-s} + \left(\frac{\delta}{2} \right)^{-s} \right] \\ &= \gamma_{s,d} - C_2 N^{\alpha(s,d)-1} - C_3 \varepsilon - C_4 N^{-1} \delta^{-s} \end{aligned} \tag{4.10}$$

with constants C_2, C_3, and C_4 independent of ε, δ, N, and x.

We begin with the proof of (3.6). Choosing $\varepsilon = \delta = N^{-1/(s+1)}$, (4.10) yields for $|x - x_j^*| = N^{-1/(s+1)}$

$$U_{i,j}(x) \ge \gamma_{s,d} - C_2 N^{\alpha(s,d)-1} - C_3 N^{-1/(s+1)} - C_4 N^{-1} N^{s/(s+1)} \ge \gamma_{s,d} - C_5 N^{-1/(s+1)}.$$

(Here, we have used the fact that, for $s < d - 1$, $1 - \alpha(s,d) > 1/(s+1)$.) Clearly, $U_{i,j}(x)$ is superharmonic in $\mathbb{R}^{d+1}$, and so we conclude that

$$U_{i,j}\left(x_j^* \right) \ge \gamma_{s,d} - C_5 N^{-1/(s+1)}.$$

Therefore

$$U_j\left(x_j^* \right) = U_{i,j}\left(x_j^* \right) + N^{-1} \left| x_i^* - x_j^* \right|^{-s} \ge \gamma_{s,d} - C_5 N^{-1/(s+1)} + N^{-1} \left| x_i^* - x_j^* \right|^{-s} \tag{4.11}$$

Combining (4.6) and (4.11), we finally get

$$\gamma_{s,d} > \gamma_{s,d} - C_5 N^{-1/(s+1)} + N^{-1} \left| x_i^* - x_j^* \right|^{-s}.$$

Hence,

$$C_5 N \left| x_i^* - x_j^* \right|^s \ge N^{1/(s+1)},$$

and (3.6) is proved. ∎

For (3.8), let $P(x_j^*)$, denote the d-dimensional hyperplane touching S^d at x_j^*. Then, if $x \in P(x_j^*)$ with $|x - x_j^*| = \delta > 0$ (i.e., x belongs to the $(d-1)$-dimensional sphere with center x_j^* and radius δ contained in

$P(x_j^*)$), then clearly $|x| < 1 + \delta^2$. Thus, choosing $\delta = N^{-1/(s+2)}$ and $\varepsilon = \delta^2 = N^{-2/(s+2)}$, we conclude from (4.10) that

$$U_{i,j}(x) \geq \gamma_{s,d} - C_2 N^{s/(s+2)-1} - C_3 N^{-2/(s+2)} - C_4 N^{-1} N^{s/(s+2)} = \gamma_{s,d} - C_6 N^{-2/(s+2)}$$

on the $(d-1)$-dimensional sphere $\{|x - x_j^*| = N^{-1/(s+2)}\} \cap P(x_j^*)$. Since $s \leq d-2$, it is easy to see that $U_{i,j}(x)$, restricted to $P(x_j^*)$, is superharmonic and, therefore,

$$U_{i,j}\left(x_j^*\right) \geq \gamma_{s,d} - C_6 N^{-2/(s+2)}.$$

Similarly to (4.11), we find that

$$U_j\left(x_j^*\right) = U_{i,j}\left(x_j^*\right) + N^{-1}\left|x_i^* - x_j^*\right|^{-s} \geq \gamma_{s,d} - C_6 N^{-2/(s+2)} + N^{-1}\left|x_i^* - x_j^*\right|^{-s},$$

which together with (4.6) yields

$$\gamma_{s,d} > \gamma_{s,d} - C_6 N^{-2/(s+2)} + N^{-1}\left|x_i^* - x_j^*\right|^{-s}.$$

Thus,

$$C_6 N \left|x_i^* - x_j^*\right|^s \geq N^{2/(s+2)},$$

and (3.8) follows. ■

4.3 Proof of Theorems 3.10 and 3.15

To simplify notations, let us denote $\rho^* := \rho_s^*(A, N)$. We can obviously assume that

$$\rho^* \geq cN^{-1/d}, \tag{4.12}$$

where c is the constant from the separation estimate (3.4).

In [14, Lemma 3.1], the authors proved a two-sided estimate on $\mathcal{E}_s(A, N)$ for the case when $A \subset \mathbb{R}^d$ is a bounded set with nonempty interior. (See also [23, Theorem 2] for the case of the unit sphere $S^d \subset \mathbb{R}^{d+1}$.) It can be easily verified that the approach used in proving the lower estimate,

$$\mathcal{E}_s(A, N) \geq c_1 N^{1+s/d}$$

with a constant $c_1 > 0$ depending on d, s, and A, but not on N, can be applied to any set $A \in \mathcal{A}^d$. Therefore, for the potential $U^{\nu_{s,N}^*}(x)$ defined in (4.1), (4.5) yields the estimate

$$U^{\nu_{s,N}^*}(x) > \frac{2}{N^2}\left(c_1 N^{1+s/d}\right) = 2c_1 N^{s/d-1}, \qquad x \in A.$$

In particular, this estimate is valid for $x = y^*$, where y^* is a point at which the maximum in (3.3) is attained (with $\boldsymbol{\omega}_N = \boldsymbol{\omega}_s^*(A, N)$). Thus

$$U^{\nu_{s,N}^*}(y^*) > 2c_1 N^{s/d-1}. \tag{4.13}$$

We now derive an upper estimate for $U^{\nu_{s,N}^*}(y^*)$. First, we note that

$$|y^* - x_i^*| \geq \rho^*, \qquad i = 1, \ldots, N.$$

Let $E_i^* := E\left(x_i^*, (c/4)N^{-1/d}\right)$. It is clear that

$$E_i^* \cap E_j^* = \emptyset, \qquad i \neq j. \tag{4.14}$$

We also have that

$$\rho^* = \min_{x \in \boldsymbol{\omega}_s^*(A,N)} |x - y^*| \leq |x_i^* - y^*|, \qquad 1 \leq i \leq N. \tag{4.15}$$

Then for any $x \in E_i^*$, $1 \le i \le N$, (4.12) and (4.15) yield

$$|x - y^*| \le |x - x_i^*| + |x_i^* - y^*| \le \frac{cN^{-1/d}}{4} + |x_i^* - y^*| \le \frac{\rho^*}{4} + |x_i^* - y^*| \le \frac{5}{4}\,|x_i^* - y^*|\,,$$

which implies that

$$|x_i^* - y^*|^{-s} \le \left(\frac{5}{4}\right)^s \min_{x \in E_i^*} \left\{|x - y^*|^{-s}\right\}, \tag{4.16}$$

and

$$|x - y^*| \ge |x_i^* - y^*| - |x - x_i^*| \ge |x_i^* - y^*| - \frac{cN^{-1/d}}{4} \ge |x_i^* - y^*| - \frac{\rho^*}{4} \ge \frac{3\rho^*}{4}. \tag{4.17}$$

Now using (4.16), (3.9), and (3.4), we get

$$\begin{aligned} |x_i^* - y^*|^{-s} &\le \left(\frac{5}{4}\right)^s \min_{x \in E_i^*} \left\{|x - y^*|^{-s}\right\} \le \left(\frac{5}{4}\right)^s \frac{1}{\mathcal{H}^d\left(E_i^*\right)} \int\limits_{E_i^*} |x - y^*|^{-s}\, d\mathcal{H}^d(x) \\ &\le \frac{C_1}{\left[(c/4)N^{-1/d}\right]^d} \int\limits_{E_i^*} |x - y^*|^{-s}\, d\mathcal{H}^d(x) \le C_2 N \int\limits_{E_i^*} |x - y^*|^{-s}\, d\mathcal{H}^d(x). \end{aligned} \tag{4.18}$$

By (4.17),

$$\bigcup_{i=1}^N E_i^* \subset A \setminus E\left(y^*, 3\rho^*/4\right) =: D.$$

Thus (4.18), (4.14), and (2.4) yield

$$\begin{aligned} U^{\nu_{s,N}^*}\left(y^*\right) &= N^{-1} \sum_{i=1}^N |x_i^* - y^*|^{-s} \le C_2 \sum_{i=1}^N \int\limits_{E_i^*} |x - y^*|^{-s}\, d\mathcal{H}^d(x) \\ &\le C_2 \int\limits_D |x - y^*|^{-s}\, d\mathcal{H}^d(x) \le C_3 \left(\frac{3\rho^*}{4}\right)^{d-s} = C_4 \left(\rho^*\right)^{d-s}. \end{aligned} \tag{4.19}$$

Combining (4.13) and (4.19), we obtain $2c_1 N^{s/d-1} \le C_4 \left(\rho^*\right)^{d-s}$, and (3.10) follows. ∎

Theorem 3.15 can be proved in a very similar way. The only difference is that, this time, we can obtain the upper estimate (4.19) without the condition (3.9) by using the mapping $\boldsymbol{\varphi}$. Indeed, let $z_i^* := \boldsymbol{\varphi}^{-1}(x_i^*)$, $t^* := \boldsymbol{\varphi}^{-1}(y^*)$. Since $\boldsymbol{\varphi}$ is bi-Lipschitz (say, with a constant L), we have

$$|x_i^* - y^*| \ge (1/L)|z_i^* - t^*|.$$

Therefore,

$$U^{\nu_{s,N}^*}(y^*) \le N^{-1} \sum_{i=1}^N \left[(1/L)|z_i^* - t^*|\right]^{-s} = L^s N^{-1} \sum_{i=1}^N |z_i^* - t^*|^{-s}. \tag{4.20}$$

We also note that

$$\min_{j \ne i} |z_j^* - z_i^*| \ge \frac{c}{L} N^{-1/d}, \tag{4.21}$$

where c is the constant from (3.4), and (see (4.15))

$$|z_i^* - t^*| \ge \frac{\rho^*}{L}, \qquad 1 \le i \le N. \tag{4.22}$$

Let B_i^* denote the ball in $\mathbb{R}^d$ with center z_i^* and radius $(cN^{-1/d})/(4L)$. It follows from (4.21) that

$$B_i^* \cap B_j^* = \emptyset, \qquad i \neq j. \tag{4.23}$$

Then, for $z \in B_i^*$, $1 \leq i \leq N$, (4.22) yields

$$|z - t^*| \leq \frac{cN^{-1/d}}{4L} + |z_i^* - t^*| \leq \frac{\rho^*}{4L} + |z_i^* - t^*| \leq \frac{5}{4} |z_i^* - t^*|,$$

which implies that

$$|z_i^* - t^*|^{-s} \leq \left(\frac{5}{4}\right)^s \min_{z \in B_i^*} \left\{|z - t^*|^{-s}\right\}, \tag{4.24}$$

and

$$|z - t^*| \geq |z_i^* - t^*| - \frac{cN^{-1/d}}{4L} \geq |z_i^* - t^*| - \frac{\rho^*}{4L} \geq \frac{3\rho^*}{4L}. \tag{4.25}$$

Using (4.24), we conclude that

$$|z_i^* - t^*|^{-s} \leq \left(\frac{5}{4}\right)^s \frac{1}{\mathcal{H}^d(B_i^*)} \int\limits_{B_i^*} |z - t^*|^{-s} \, d\mathcal{H}^d(z) \leq C_1 N \int\limits_{B_i^*} |z - t^*|^{-s} \, d\mathcal{H}^d(z). \tag{4.26}$$

By (4.25),

$$\bigcup_{i=1}^{N} B_i^* \subset \mathbb{R}^d \setminus B^d(t^*, 3\rho^*/(4L)) =: D.$$

Thus (4.20), (4.26), and (4.23) yield

$$\begin{aligned} U^{\nu^*_{s,N}}(y^*) &\leq L^s N^{-1} \sum_{i=1}^{N} |z_i^* - t^*|^{-s} \leq C_1 L^s \sum_{i=1}^{N} \int\limits_{B_i^*} |z - t^*|^{-s} \, d\mathcal{H}^d(z) \\ &\leq C_1 L^s \int\limits_{D} |z - t^*|^{-s} \, d\mathcal{H}^d(z) = C_2 \left(\frac{3\rho^*}{4L}\right)^{d-s} = C_3 (\rho^*)^{d-s}. \end{aligned}$$

Combining this estimate with (4.13), we obtain (3.10). ∎

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