# Converse Marcinkiewicz-Zygmund inequalities on the real line with applications to mean convergence of Lagrange interpolation.

S. B. Damelin[1], H. S. Jung[2] and K. H. Kwon[2]

Recieved

**Abstract**

We investigate converse Marcinkiewicz-Zygmund inequalities on the real line and illustrate how we may use these inequalities to deduce new results on boundedness and mean convergence of Lagrange interpolation.

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## 1 Introduction

Let

$$\chi_n := \{\xi_{1,n}, \xi_{2,n}, \ldots, \xi_{n,n}\},\ n \geq 1$$

be a triangular array of points on the real line. In this paper, we prove converse weighted Marcinkiewicz-Zygmund inequalities. Such inequalities have the form

$$\int |P|^p(x)w(x)dx \leq C\sum_{j=1}^{n} \lambda_{j,n} \left|P(x_{j,n})V(x_{j,n})\right|^p, \quad P \in \mathcal{P}_{n-1},\, p > 0$$

with $\mathcal{P}_{n-1}$ the class of polynomials of degree at most $n-1$, $n \geq 1$, $C$ a positive constant independent of $P$ and $n$, $w$ and $V$ suitable positive weights and $\lambda_{jn}$ the Cotes numbers with respect to $w$ and the array $\chi_n$, see [9]. Such inequalities are in particular, useful in proving theorems on mean convergence of Lagrange interpolation. Let us recall that that if $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, then denoting by $\ell_{j,n}(\chi_n) \in \Pi_{n-1},\ 1 \le j \le n$ the fundamental polynomials of Lagrange interpolation satisfying for $1 \le k \le n$

$$\ell_{j,n}(\chi_n)(\xi_{j,n}) = \begin{cases} 1, & j = k \\ 0, & \text{otherwise,} \end{cases}$$

the **Lagrange interpolation polynomial of degree** $n-1$ to $f$ with respect to $\chi_n$, $L_n(f, \chi_n)$, admits the representation

$$L_n(f, \chi_n)(x) := \sum_{j=1}^{n} f(\xi_{j,n})\ell_{j,n}(\chi_n)(x),\ x \in \mathbb{R}.$$

One method to prove weighted Marcinkiewicz-Zygmund inequalities uses mean convergence of orthonormal expansions, see [33], [34], [35], [20], [21], [22] and the references cited therein. Another method, originally due to Kőnig for the Hermite weight, see [10], has also been studied rather extensively recently and was first adapted for a general class of Erdős weights on the real line by the first author and D.S. Lubinsky in [7]. Subsequently, the method was applied to other classes of weights by D. S. Lubinsky in [14], [15] and [16]. In particular, in the interesting paper [16], D. S. Lubinsky has recently proved converse Marcinkiewicz-Zygmund inequalities using the method of Kőnig for rather general arrays on $[-1, 1]$ and weights with applications to mean convergence of Lagrange interpolation for a class of exponential weights on $[-1, 1]$. For the real line, the situation is typically more difficult and it is the aim of this paper to investigate several new Marcinkiewicz-Zygmund inequalities for a class of fast decaying weights on the real line adapting the method of Kőnig above. We then use these inequalities to prove new theorems on mean convergence of Lagrange and extended Lagrange interpolation extending earlier work of [2] and [3]. In this paper, we choose as our class of weights on the real line, a class of even Erdős weights which are of faster than polynomial growth at infinity. We emphasize that we wish to describe our technique rather than dwell on technical assumptions on our weight class. To this end, we mention that we believe our methods apply to more general classes of weights such as even Freud weights and/or even weights with less smoothness but we do not pursue this later investigation here. Our class of weights includes as examples, see [13], the weights

$$w_{k,\alpha}(x) := \exp\big(-Q_{k,\alpha}(x)\big) \tag{1.1}$$

where

$$Q_{k,\alpha}(x) := \exp_k(|x|^\alpha),\ k \ge 1, \alpha > 1$$

and

$$w_{A,B}(x) := \exp\big(-Q_{A,B}(x)\big) \tag{1.2}$$

where

$$Q_{A,B}(x) = \exp\left(\log(A+x^2)\right)^B, \ B > 1.$$

Here, $\exp_k(;) = \exp\left(\exp(\exp(;))\right)$ denotes the $k$th iterated exponential and $A$ is a large enough but fixed absolute constant.

More precisely, we shall treat weights of the form $w^2 := e^{-2Q}$ where:

- $Q : \mathbb{R} \to [0, \infty)$ is even and continuous.
- $Q^{(2)}$ exists and $Q^{(j)}$, $j = 0, 1, 2$ is non negative in $(0, \infty)$.
- The function

  $$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}$$

  is increasing in $(0, \infty)$ with

  $$\lim_{x\to\infty} T(x) = \infty,$$

  and

  $$T(0^+) := \lim_{x\to 0^+} T(x) > 1.$$

- There exists $C > 0$ such that for sufficiently large $x$

  $$\frac{1}{C} \le \frac{T(x)}{\frac{xQ'(x)}{Q(x)}} \le C.$$

- For every $\epsilon > 0$, there exists a positive constant $A_1$ so that uniformly for $|x| \ge C$,

  $$T(x) \le A_1 (Q(x))^{\epsilon}.$$

Such weights, see [13] and [7] will be denoted by $\mathcal{E}$.

# Interpolation points

Given $w \in \mathcal{E}$, see [9], we may define orthonormal polynomials

$$p_n(x) := p_n(w^2, x) = \gamma_n x^n + \cdots, \ \gamma_n = \gamma_n(w^2) > 0, \ x \in \mathbb{R}$$

satisfying

$$\int_{\mathbb{R}} p_n(w^2, x) p_m(w^2, x) w^2(x) dx = \begin{cases} 0, & n \ne m \\ 1, & n = m \end{cases}$$

and with zeros denoted by

$$-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty.$$

For each $n \geq 1$ and for the given weight $w$, we define interpolatory matrices

$$U_n := \{x_{j,n} : 1 \leq j \leq n\} \tag{1.3}$$

and

$$V_{n+2} = U_n \cup \{y_0\} \cup \{-y_0\} \tag{1.4}$$

where $y_0$ maximizes $\|p_n w\|_{L_\infty(\mathbb{R})}$. It is important to note that $\pm y_0 \notin U_n$.

The existence of $y_0$ follows from the Mhaskar-Rakhmanov-Saff identity, see ([30], Theorem 3.2.1),

$$\|Pw\|_{L_\infty(\mathbb{R})} = \|Pw\|_{L_\infty[-a_n,a_n]}$$

valid for every $P \in \Pi_n$, $n \geq 1$ where the number $a_u$ is defined for every $u > 0$ as the positive root of the equation

$$u := \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)dt}{\sqrt{1-t^2}}.$$

Defining for $x \in \mathbb{R}$

$$\tilde{p}_n(x) := p_n(x)(x - y_0)(x + y_0),$$

we define

$$l_{j,n+2}(V_{n+2})(x) = \frac{\tilde{p}_n(x)}{\tilde{p}_n'(x_{jn})(x - x_{jn})} \quad \text{for } j = 0, \cdots, n+1$$

and let

$$\begin{aligned} L_{n+2}(f, V_{n+2})(x) &= \sum_{j=1}^{n} f(x_{j,n})\ell_{j,n+2}(V_{n+2})(x) \\ &+ f(y_0)\ell_{n+1,n+2}(V_{n+2})(x) + f(-y_0)\ell_{n+2,n+2}(V_{n+2})(x) \end{aligned}$$

denote the extended Lagrange interpolation polynomial of degree at most $n+2$ to $f$ where we may consider $x_{0n} := y_0$ and $x_{n+1,n} := -y_0$.

# 2 Converse Quadrature Formulae

## 2.1 Converse quadrature formula with fixed weight $(1 + |x|)^r$

In this section, we present two converse quadrature formulae from which we deduce theorems on Lagrange interpolation. We begin with:

**Theorem 2.1** *Let $w \in \mathcal{E}$, $1 < p < 4$ and $R, r \in \mathbb{R}$. Moreover assume that*

$$r < 1 - 1/p,\ r \leq R,\ R > -1/p \tag{2.1}$$

*and let $\lambda_{j,n} = \lambda_n(w^2,\ x_{j,n})$, $1 \leq j \leq n$ be the Cotes numbers for the weight $w^2$. Then there exists a positive constant $C$ such that for every polynomial $P \in \mathcal{P}_{n-1}$*

$$\begin{aligned} &\|Pw(x)(1+|x|)^r\|_{L_p(\mathbb{R})} \qquad (2.2)\\ &\leq C \quad \left\{ \sum_{j=1}^{n} \lambda_{jn}\, |Pw(x_{jn})|^p\, w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp} \right\}^{1/p}. \end{aligned}$$

**Theorem 2.2** *Let $w \in \mathcal{E}$, $4 < p < \infty$ and $R, r \in \mathbb{R}$. Moreover assume (2.1) and let $\lambda_{j,n} = \lambda_n(w^2,\ x_{j,n})$, $1 \leq j \leq n$ be the Cotes numbers for the weight $w^2$. Then there exists a positive constant $C$ such that for every polynomial $P \in \mathcal{P}_{n+1}$*

$$\begin{aligned} &\|Pw(x)(1+|x|)^r\|_{L_p(\mathbb{R})} \qquad (2.3)\\ &\leq C \quad \left\{ \sum_{j=1}^{n} \lambda_{jn}\, |Pw(x_{jn})|^p\, w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp} \right\}^{1/p} + B, \end{aligned}$$

*where for some $s > 1$*

$$\begin{aligned} B: \quad = \quad & \|Pw(y_0)(1+|x|)^R\|_{L_p(|x| \leq a_{sn})} + \|Pw(-y_0)(1+|x|)^R\|_{L_p(|x| \leq a_{sn})} \\ + \quad & \sup_{|x_{jn}| \geq 2a_n/3} |Pw(x_{jn})| \left\| (1+|x|)^R \right\|_{L_p(|x| \leq a_n/2)}. \qquad (2.4) \end{aligned}$$

## Lagrange interpolation

As consequences of Theorems 2.1 and 2.2, we are able to deduce the following corollaries on mean convergence of Lagrange interpolation.

**Corollary 2.3** *Let $w \in \mathcal{E}$, $0 < p < 4$, $\Delta \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $\hat{\alpha} := \min\{\alpha, 1\}$. Then if $\alpha > 0$, for*

$$\lim_{n \to \infty} \left\| \left(L_n(f, U_n) - f\right)(x) w(x)(1+|x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} = 0, \tag{2.5}$$

*to hold for every continuous function $f$ satisfying*

$$\lim_{|x| \to \infty} f w(x)(1+|x|)^{\alpha} = 0 \tag{2.6}$$

*it is sufficient that*

$$\hat{\alpha} + \Delta > 1/p. \tag{2.7}$$

*Moreover, for (2.5) to hold for every continuous function $f$ satisfying (2.6), it is necessary that (2.7) holds.*

Corollary 2.3 is interesting in that it includes the necessary and sufficient conditions of Theorem 1.2 and Theorem 1.3 of [8] as special cases. For $p > 4$, the first author and D. S. Lubinsky showed in [7], that a weight factor which decays as a power of $1 + Q$ was necessary and sufficient for convergence under the same decay conditions on $fw$ given by (2.6). Our next Corollary shows that replacing $U_n$ in (2.5) by $V_{n+2}$, allows us to extend the range of $p$ in Corollary 2.3 to $0 < p < \infty$.

**Corollary 2.4** *Let $w \in \mathcal{E}$, $0 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $\hat{\alpha} := \min\{\alpha, 1\}$ . Then if $\alpha > 0$, for*

$$\lim_{n\to\infty} \left\| \left(L_{n+2}(f, V_{n+2}) - f\right)(x)w(x)(1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} = 0 \tag{2.8}$$

*to hold for every continuous function $f$ satisfying* (2.6) *it is sufficient that* (2.7) *holds. Moreover for* (2.8) *to hold for every continuous function $f$ satisfying* (2.6) *it is necessary that* (2.7) *holds.*

Corollary 2.4 shows that under a polynomial decay condition on $fw$, a polynomial decay term is necessary and sufficient for weighted convergence in $L_p(0 < p < \infty)$ with respect to the interpolatory matrix $V_{n+2}$. That means that for Erdős weights, mean convergence of extended Lagrange interpolation is almost the same as mean convergence of Lagrange interpolation in $L_p(0 < p < 4)$ with respect to $U_n$. Finally we mention that in [5], theorems on extended Lagrange interpolation for $0 < p < \infty$ are proved but under a logarithmic decay condition of $fw$. This typically forces stronger weighting factors than the ones above but still weaker ones to those in [7] where interpolation with respect to $U_n$ is studied.

## 2.2 Converse quadrature formulae with $(|1 - |x|/a_n| + L\delta_n)^\beta$

In this section, we record a converse quadrature inequality but with varying weight $(|1 - |x|/a_n| + L\delta_n)^\beta$ for some natural exponent $\beta$ and state some useful corollaries on boundedness and mean convergence of Lagrange interpolation. Our choice of weight is influenced by the fact that close to $a_n$, this factor behaves as a power of $n$ which is sufficient to conclude mean convergence of Lagrange interpolation with respect to $U_n$ for $1 < p < \infty$ over the entire real line. We do not provide proofs but refer the reader to the papers [2] and [15] where the details may be found.

**Theorem 2.5** *Let $w \in \mathcal{E}$, $1 < p < \infty$ and let $\lambda_{j,n} = \lambda_n(w^2, x_{j,n})$, $1 \le j \le n$ be the Cotes numbers for the weight $w^2$. Let $\beta$ satisfy*

$$1/4 - 1/p < \beta < \min\{5/4 - 1/p, 3/4 + 1/2p\}. \tag{2.9}$$

*Then there exists a positive constant $C$ such that uniformly for $n \geq 1$ and $P \in \Pi_{n-1}$*

$$\left\| Pw(x) \left( \left| 1 - \frac{|x|}{a_n} \right| + Ln^{-2/3} T(a_n)^{-2/3} \right)^{\beta} \right\|_{L_p(\mathbb{R})} \tag{2.10}$$
$$\leq \quad C \left\{ \sum_{j=1}^{n} \lambda_{j,n} w^{-2}(x_{j,n}) |Pw|^p (x_{j,n}) \left( \left| 1 - \frac{|x_{j,n}|}{a_n} \right| + Ln^{-2/3} T(a_n)^{-2/3} \right)^{\beta p} \right\}^{\frac{1}{p}}.$$

A special case of Theorem 2.5 was first proved for $1 < p < 4$ in ([8], Theorem 3.1) although an earlier formulation for the Hermite weight appeared first in ([10], Theorem 1(a)) for $1 < p < 4$. (2.10) has also been established for $1 < p < 4$ for a class of Freud weights by D. S. Lubinsky in ([14], Theorem 1.2) and for $p \geq 1$ for a class of exponential weights on $(-1, 1)$ by D. S. Lubinsky in ([15], Theorem 1.2). We observe that the parameter $\beta$ in (2.9) may be taken as 0 for $1 < p < 4$. This is consistent with the results of [8]. One of the main ideas in Theorem 2.5, in our opinion, is that it explains why no weighting factor is needed for $1 < p < 4$, while at the same time, it illustrates that a weighting factor, which is really only significant near $a_n$, is also sufficient for $p \geq 4$. This weighting factor actually grows as a power of $n$ near $a_n$ much as $Q(a_n)$ does. The significance of this observation is important and is discussed further after the statement of Corollary 2.6.

As a consequence of Theorem 2.5, we record:

**Corollary 2.6** *Let $w \in \mathcal{E}$, $p > 1$, $\alpha > 0$ and $\beta$ as in* (2.9). *Further let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with*

$$\lim_{|x| \longrightarrow \infty} |f(x)| \, w(x) \, (1 + |x|)^{\alpha} = 0.$$

*Then given $\delta > 0$, there exists a positive constant $C$, such that for all $n \geq C$*

$$\left\| (f - L_n(f, U_n))(x) w(x) \left( \left| 1 - \frac{|x|}{a_n} \right| + Ln^{-2/3} T(a_n)^{-2/3} \right)^{\beta} \right\|_{L_p(\mathbb{R})} \tag{2.11}$$
$$\leq \delta \left\| (1 + |x|)^{-\alpha} \left( \left| 1 - \frac{|x|}{a_n} \right| + Ln^{-2/3} T(a_n)^{-2/3} \right)^{\beta} \right\|_{L_p(|x| \leq a_{2n})}.$$

Thus if $1 < p < 4$, we may take $\beta = 0$ and the right hand side of (2.11) can be made arbitrary small, if $\alpha > 1/p$. Moreover, if $p \geq 4$ and then $\beta > 0$, the right hand side of (2.11) is arbitrary small if $\alpha > 1/p$ and the factor on the left hand side of (2.11) grows as a power of $n$, (see (4.9) below), close to $a_n$. As

$Q(a_n)$ also grows as a power of $n$, (see (4.10) below), this explains why a factor $1+Q$ was necessary in Theorem 1.3 of [7]. The factor

$$\left(\left|1-\frac{|x|}{a_n}\right| + Ln^{-2/3}T(a_n)^{-2/3}\right)^{\beta}$$

is in a sense more natural, as for $p \geq 4$, it is significant near $a_n$ as it should be and is identically 1 for $1 < p < 4$. This is also consistent with the results of ([8], Theorem 1.3).

## Unboundedness of $\{L_n\}$

Finally we deduce results on the unboundedness of $\{L_n\}$ as a sequence of operators from weighted $L_p$ to weighted $L_\infty$. Corresponding results from weighted $L_\infty$ to $L_\infty$ may be found in [3] and [4] and similar results from weighted $L_p$ to $L_p$ may be deduced from the results of [11].

**Corollary 2.7** *Let $w \in \mathcal{E}$, $1 < p < \infty$, $n \geq 1$ and $\beta$ as in (2.9). Then for every $f : \mathbb{R} \to \mathbb{R}$ continuous, there exists a constant $C$ independent of $f$ and $n$ such that*

$$\left\| L_n(f, U_n)(x)w(x)\left(\left|1-\frac{|x|}{a_n}\right| + Ln^{-2/3}T(a_n)^{-2/3}\right)^{\beta}\right\|_{L_p(\mathbb{R})} \tag{2.12}$$
$$\leq C\|fw\|_{L_\infty(\mathbb{R})}\left\|\left(\left|1-\frac{|x|}{a_n}\right| + Ln^{-2/3}T(a_n)^{-2/3}\right)^{\beta}\right\|_{L_p(|x|\leq a_{2n})}.$$

*In particular, if $1 < p < 4$*

$$\|L_n(f,U_n)w\|_{L_p(\mathbb{R})} \leq Ca_n^{1/p}\|fw\|_{L_\infty(\mathbb{R})}$$

*and if $p \geq 4$*

$$\left\| L_n(f, U_n)(x)w(x)\left(\left|1-\frac{|x|}{a_n}\right| + Ln^{-2/3}T(a_n)^{-2/3}\right)^{\beta}\right\|_{L_p(\mathbb{R})} \leq Ca_n^{1/p}\|fw\|_{L_\infty(\mathbb{R})}.$$

**Remark 2.8**

(a) The appearance of the factor $a_n^{1/p}$ in Corollary 2.7 is unfortunate but necessary. This is illustrated by the following result which follows easily using the methods of ([8], Theorem 1.3).

*Let $w \in \mathcal{E}$ and $1 < p < 4$. There exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ with*

$$\lim_{|x|\to\infty} |gw|(x) = 0$$

*and satisfying for some positive constant $C$*

$$\|L_n(g,U_n)w\|_{L_p(\mathbb{R})} \geq Ca_n^{1/p}\|gw\|_{L_\infty(\mathbb{R})}. \tag{2.13}$$

(b) Recalling that $a_n$ increases with $n$ we see that even with an extra weighting factor depending on $n$ in the left hand side of (2.12), the weighted Lagrange operator is unbounded as a sequence of operators from $L_p$ to $L_\infty$ and thus decay conditions on $f$ are necessary to ensure mean convergence.

Section 3, is devoted to the proofs of Theorems 2.1, 2.2 and their corollaries.

# 3 Proofs of Results

We begin with the proof of Theorem 2.2. The main idea of the proof, due to Kőnig, is to write the Lagrange interpolation polynomial partly as a discrete Hilbert transform. To this end, we find it convenient to break up the proof of Theorem 2.2 into several auxiliary lemmas following the format of [10] and [8]. We will often need information regarding $p_n(w^2)$ and its zeros and for this we will refer to the paper [7] where this information is readily available. The estimates themselves appeared first in [13].

We find it convenient to fix the following notation which will be used throughout.

Firstly, for any two sequences $(b_n)$ and $(c_n)$ of nonzero real numbers, we shall write

$$b_n \lesssim c_n,$$

if there exists a constant $C > 0$, independent of $n$ such that $b_n \leq Cc_n$ for $n$ large enough and write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. Similar notation will be used for functions and sequences of functions.

Next we set:

(a)

$$I_{jn} := (x_{jn}, x_{j-1,n}) \text{ and } |I_{jn}| := x_{j-1,n} - x_{jn}, \quad 1 \leq j \leq n.$$

(b)

$$\chi_{jn} := \chi_{I_{jn}} := \text{ the indicator function of } I_{jn}, \quad 1 \leq j \leq n.$$

(c)

$$\Psi_b(x) := (1 + |x|)^b, \qquad \text{for } b, x \in \mathbb{R}.$$

(d)

$$\zeta_n(x) := \left(|1 - |x|/a_n| + (nT(a_n))^{-2/3}\right)^{1/4}, \, x \in \mathbb{R}.$$

(e)

$$\tilde{p}_n(x) := p_n(x)(x - y_0)(x + y_0).$$

Following the format of [10], we break up the proof of Theorem 2.2 into several auxiliary lemmas.

## Bound for the Hilbert Transform

We recall that given $g \in L_1(\mathbb{R})$, the Hilbert transform of $g$ exists a.e and is denoted by

$$H[g](x) := \lim_{\varepsilon \to 0^+} \int_{|t-x| \geq \varepsilon} \frac{g(t)}{x - t} dt.$$

The following lemma has not appeared in the literature and so we state it and provide a short proof.

**Lemma 3.1** *Let $b, B \in \mathbb{R}$, $K \geq 2$, $n \geq C$ and $p > 4$. Assume that*

$$b < 1 - 1/p, \quad b \leq B \text{ and } B > -1/p.$$

*Then for $g \in L_p(-Ka_n, Ka_n)$,*

$$\begin{aligned} &\left\| H[g](x)\Psi_b(x) \left|1 - |x|/a_n\right|^{3/4} \right\|_{L_p(a_n/2 \leq |x| \leq Ka_n)} \\ \lesssim &\left\| g(x)\Psi_B(x) \left|1 - |x|/a_n\right|^{3/4} \right\|_{L_p(|x| \leq Ka_n)}. \end{aligned} \tag{3.1}$$

**Proof** We write $g = g_1 + g_2$ where $g_1$ vanishes outside $[-a_n/4, a_n/4]$ and $g_2$ vanishes inside $[-a_n/4, a_n/4]$. Firstly, it is a well known result of Riez that $H$ is a bounded operator from $L_p(1 < p < \infty)$ to $L_p(1 < p < \infty)$. Indeed, the following modification is also true and follows easily using [15, Lemma 3.1] which follows from ideas in [25].

For $f \in L_p(\mathbb{R})$ and $1 < p < \infty$,

$$\begin{aligned} &\left\| H[g](x) \left( \left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}T(a_n)^{-2/3} \right)^b \right\|_{L_p(-Ka_n, Ka_n)} \\ \lesssim &\left\| g(x) \left( \left|1 - \frac{|x|}{a_n}\right| + n^{-2/3}T(a_n)^{-2/3} \right)^b \right\|_{L_p(-Ka_n, Ka_n)}. \end{aligned} \tag{3.2}$$

Then applying (3.2) and recalling that $p > 4$ gives

$$\begin{aligned} &\|H[g_2](x)\Psi_b(x) \left|1 - |x|/a_n\right|^{3/4} \|_{L_p(a_n/2 \leq |x| \leq Ka_n)} \\ \lesssim \;& a_n^b \|H[g_2](x) \left|1 - |x|/a_n\right|^{3/4} \|_{L_p(a_n/2 \leq |x| \leq Ka_n)} \\ \lesssim \;& \|g(x)\Psi_B(x) \left|1 - |x|/a_n\right|^{3/4} \|_{L_p(a_n/4 \leq |x| \leq Ka_n)}. \end{aligned}$$

Finally recalling the identity, cf ([26], Lemma 8, pg 440),

$$\|H[g]\Psi_b\|_{L_p(\mathbb{R})} \lesssim \|g\Psi_B\|_{L_p(\mathbb{R})},$$

allows us to write

$$\begin{aligned} & \|H[g_1](x)\Psi_b(x)\,|1-|x|/a_n|^{3/4}\,\|_{L_p(a_n/2\leq|x|\leq Ka_n)} \\ \lesssim\ & \|g\Psi_B(x)\|_{L_p(|x|\leq a_n/4)} \lesssim \|g\Psi_B(x)\,|1-|x|/a_n|^{3/4}\,\|_{L_p(|x|\leq a_n/4)}. \end{aligned}$$

Combining our two identities for $g_1$ and $g_2$ then gives the lemma. □

## Operator inequality

We now state an operator inequality of Kőnig, see [8, Lemma 2.5, p. 745].

**Lemma 3.2** *Let $1 < p < \infty$ and $q := \frac{p}{(p-1)}$. Let $(\Omega, \mu)$ be a measure space, $k, r : \Omega^2 \longrightarrow \mathbb{R}$ and set*

$$T_k[f](u) := \int_\Omega k(u,v)f(v)d\mu(v)$$

*for $\mu$ measurable $f : \Omega \longrightarrow \mathbb{R}$ . Assume that,*

$$\sup_u \int_\Omega |k(u,v)|\,|r(u,v)|^q\,d\mu(v) \leq M$$

*and*

$$\sup_v \int_\Omega |k(u,v)|\,|r(u,v)|^{-p}\,d\mu(u) \leq M.$$

*Then $T_k$ is a bounded operator from $L_p(d\mu)$ to $L_p(d\mu)$.*

## Replacement of $H[\chi I_{jn}]$

Next we record a lemma whereby we replace the fraction $\frac{1}{x-x_{j,n}}$ by $H[\chi I_{jn}]$ for every $1 \leq j \leq n$. The idea appeared first in [10] and the proof is very similar to that of [8].

**Lemma 3.3** *Uniformly for $n \geq 1$, $1 \leq j \leq n$, $x \in [x_{nn}, x_{1n}]$ and fixed $R \in \mathbb{R}$,*

$$\begin{aligned} \tau_{jn}(x) = \tau_{jn} \quad &:= \left|\frac{\Psi_{-R}(x_{jn})}{x - x_{jn}} - \frac{1}{|I_{jn}|}H[\chi_{jn}\Psi_{-R}](x)\right| a_n^{1/2}|\tilde{p}_n w|(x) \qquad (3.3) \\ &\lesssim a_n^2 f_{jn}(x)\zeta_n^3(x) \end{aligned}$$

*where $f_{jn}$ is defined by*

$$f_{jn}(x) := \Psi_{-R}(x_{jn}) \begin{cases} |I_{jn}|^{-1} & \textit{if } |x - x_{jn}| \leq 2|I_{jn}| \\ \frac{|I_{jn}|}{|x-x_{jn}|}\left(\frac{1}{|x-x_{jn}|} + \frac{1}{1+|x_{jn}|}\right) & \textit{if } |x - x_{jn}| > 2|I_{jn}|. \end{cases}$$

We are now in a position to present:

**The Proof of Theorem 2.2** We first note that it is sufficient to prove that

$$\begin{aligned} & \left\|L_{n+2}(P,V_{n+2})(x)w(x)(1+|x|)^R\right\|_{L_p(\mathbb{R})} \\ \lesssim & \left\{\sum_{j=1}^{n}\lambda_{jn}\left|Pw(x_{jn})\right|^p w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp}\right\}^{1/p} + B \end{aligned}$$

where B is given by (2.4).

**Step 1:** Express $L_{n+2}(P,V_{n+2})(x)w(x)(1+|x|)^R$ as a sum of two terms.

Let us write for $x \in \mathbb{R}$

$$L_{n+2}(P,V_{n+2})(x)w(x)(1+|x|)^R = A_1(x) + A_2(x)$$

where

$$A_1(x) := \sum_{j=1}^{n} P(x_{jn})l_{j,n+2}(V_{n+2})(x)w(x)(1+|x|)^R$$

and

$$\begin{aligned} & A_2(x) := P(y_0)l_{0,n+2}(V_{n+2})(x)w(x)(1+|x|)^R + \\ & +P(-y_0)l_{n+1,n+2}(V_{n+2})(x)w(x)(1+|x|)^R. \end{aligned}$$

Firstly, we estimate $A_2(x)$. It suffices to assume that $|x| \le a_{sn}$ for some $s > 1$. Observe that for this range of $|x|$, we have using [3, (2.12)]

$$\begin{aligned} & \left|P(y_0)w(x)l_{n+1,n+2}(x)(1+|x|)^R\right| \\ = & \left|P(y_0)w(y_0)(1+|x|)^R \frac{p_n(x)w(x)(x+y_0)(x-y_0)}{p_n(y_0)w(y_0)(2y_0)(x-y_0)}\right| \\ \lesssim & |Pw|(y_0)(1+|x|)^R \left|\frac{(x+y_0)}{a_n}\right| \\ \lesssim & |Pw|(y_0)(1+|x|)^R. \end{aligned}$$

Similarly estimating the second term in $A_2$, we see that

$$\|A_2(x)\|_{L_p(|x|\le a_{sn})} \lesssim B. \tag{3.4}$$

**Step 2:** Express $A_1$ as a sum of two terms.

$$\begin{aligned} A_1(x) &= \sum_{j=1}^{n} P(x_{jn})l_{j,n+2}(x)w(x)(1+|x|)^R \\ &= a_n^{1/2}\tilde{p}_n w(x)(1+|x|)^R \sum_{j=1}^{n} P(x_{jn})w(x_{jn}) \frac{a_n^{-1/2}}{\tilde{p}_n'(x_{jn})w(x_{jn})(x-x_{jn})} \\ &= a_n^{1/2}\tilde{p}_n w(x)(1+|x|)^R \sum_{j=1}^{n} y_{jn} \frac{\Psi_{-R}(x_{jn})}{(x-x_{jn})} \end{aligned}$$

where $y_{jn}$ is defined by

$$y_{jn} := \frac{a_n^{-1/2} P(x_{jn}) w(x_{jn})}{\tilde{p}_n'(x_{jn}) w(x_{jn})} \Psi_R(x_{jn})$$

for $1 \leq j \leq n$. We observe that using [7, (2.11)] below and ([3], (2.24)), we have

$$|y_{jn}| \sim |P(x_{jn}) w(x_{jn}) \Psi_R(x_{jn})| \frac{|I_{jn}| \zeta_n^{-3}(x_{jn})}{a_n^2}. \tag{3.5}$$

**Step 2-A:** Now, we will estimate $A_1(x)$. It suffices to consider two cases. $|x| \leq a_n/2$ and $a_n/2 \leq |x| \leq 2a_n$. We begin with the case $a_n/2 \leq |x| \leq 2a_n$. Now let us write

$$\begin{aligned} A_1(x) \quad &= \quad a_n^{1/2} \tilde{p}_n w(x)(1+|x|)^R \sum_{j=1}^n y_{jn} \left\{ \frac{\Psi_{-R}(x_{jn})}{x - x_{jn}} - \frac{1}{|I_{jn}|} H[\chi_{jn} \Psi_{-R}](x) \right\} \\ &\quad + a_n^{1/2} \tilde{p}_n w(x)(1+|x|)^R H \left[ \sum_{j=1}^n \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \Psi_{-R} \right](x) \\ &:= \quad J_1(x) + J_2(x) \end{aligned}$$

**Step 3-A:** Estimate $\|J_2(x)\|_{L_p(a_n/2 \leq |x| \leq 2a_n)}$.
Since

$$\int_{I_{jn}} |1 - |x|/a_n|^{3p/4} \, dx \sim \zeta_n^{3p}(x_{jn}) |I_{jn}|,$$

we have using Lemma 3.1 and (3.5),

$$\begin{aligned} & \|J_2(x)\|_{L_p(a_n/2 \leq |x| \leq 2a_n)} \\ \lesssim \quad & a_n^2 \left\| |1 - |x|/a_n|^{3/4} \, \Psi_R(x) H \left[ \sum_{j=1}^n \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \Psi_{-R} \right](x) \right\|_{L_p(a_n/2 \leq |x| \leq 2a_n)} \\ \lesssim \quad & a_n^2 \left\| |1 - |x|/a_n|^{3/4} \, \Psi_R(x) \sum_{j=1}^n \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \Psi_{-R}(x) \right\|_{L_p[-2a_n, 2a_n]} \\ = \quad & a_n^2 \left\{ \sum_{j=1}^n \left( \frac{y_{jn}}{|I_{jn}|} \right)^p \int_{I_{jn}} |1 - |x|/a_n|^{3p/4} \, dx \right\}^{1/p} \\ \sim \quad & \left\{ \sum_{j=1}^n |Pw(x_{jn}) \Psi_R(x_{jn})|^p \, |I_{jn}| \right\}^{1/p} . \end{aligned} \tag{3.6}$$

**Step 4-A:** Estimate $\|J_1(x)\|_{L_p(a_n/2 \leq |x| \leq 2a_n)}$.

By Lemma 3.3

$$|J_1(x)| \lesssim a_n^2 \sum_{j=1}^{n} |y_{jn}| f_{jn}(x)\zeta_n^3(x)(1+|x|)^R, \quad x \in [x_{nn}, x_{1n}].$$

Thus

$$\begin{aligned}
& \|J_1(x)\|_{L_p(a_n/2 \le |x| \le x_{1n})} \\
\lesssim \quad & a_n^2 \left\{ \sum_{|x_{kn}| \ge a_n/2} \left[ \int_{I_{kn}} \sum_{j=1}^{n} |y_{jn}| f_{jn}(x)\zeta_n^3(x)(1+|x|)^R \right]^p dx \right\}^{1/p} \\
\lesssim \quad & a_n^2 \left\{ \sum_{|x_{kn}| \ge a_n/2} |I_{kn}| \left[ \sum_{j=1}^{n} |y_{jn}| f_{jn}(x_{kn})\zeta_n^3(x_{kn})\Psi_R(x_{kn}) \right]^p \right\}^{1/p}.
\end{aligned}$$

Here we have used the fact that $f_{jn}$ does not change much in $I_{kn}$. Using the definition of $f_{jn}$, we see that

$$\|J_1(x)\|_{L_p[a_n/2 \le |x| \le x_{1n}]} \lesssim (S_1 + S_2 + S_3),$$

where:

$$\begin{aligned}
S_1 \quad := \quad & a_n^2 \left\{ \sum_{|x_{kn}| \ge a_n/2} |I_{kn}| \left[ \sum_{j=1, j \ne k} |y_{jn}| \Psi_{-R}(x_{jn}) \frac{|I_{jn}|}{(x_{kn} - x_{jn})^2} \zeta_n^3(x_{kn})\Psi_R(x_{kn}) \right]^p \right\}^{1/p}, \\
S_2 \quad := \quad & a_n^2 \left\{ \sum_{|x_{kn}| \ge a_n/2} |I_{kn}| \left[ \sum_{j=1, j \ne k} |y_{jn}| \Psi_{-R}(x_{jn}) \frac{|I_{jn}|}{|x_{kn} - x_{jn}|(1+|x_{jn}|)} \zeta_n^3(x_{kn})\Psi_R(x_{kn}) \right]^p \right\}^{1/p}
\end{aligned}$$

and

$$S_3 := a_n^2 \left\{ \sum_{|x_{kn}| \ge a_n/2} |I_{kn}| \left[ \frac{y_{kn}}{|I_{kn}|} \Psi_{-R}(x_{kn})\zeta_n^3(x_{kn})\Psi_R(x_{kn}) \right]^p \right\}^{1/p}.$$

**Step 5-A:** Estimate $S_j$, $j = 1, 2, 3$.

Firstly we see that we may easily deduce that

$$S_3 \lesssim \left\{ \sum_{k=2}^{n} |I_{kn}| \, |Pw(x_{kn})\Psi_R(x_{kn})|^p \right\}^{1/p}. \tag{3.7}$$

Next,

$$S_1 \lesssim \left\{ \sum_{k=1}^{n} \left[ \sum_{j=1}^{n} b_{kj} |I_{jn}|^{1/p} |Pw(x_{jn})| \Psi_R(x_{jn}) \right]^p \right\}^{1/p},$$

where,

$$b_{k,k} := 0 = b_{1,k} \text{ for every } k$$

and for $j \neq k$,

$$\begin{aligned} b_{k,j} \quad := \quad & \chi_{|x_{kn}| \geq a_n/2} \left|I_{j,n}\right|^{2-1/p} \left|I_{k,n}\right|^{1/p} (x_{j,n} - x_{k,n})^{-2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \\ & \times \Psi_{-R}(x_{jn}) \Psi_R(x_{kn}). \end{aligned}$$

Here $\chi_{S(k)}$ equals 1 if for $k$, $S(k)$ holds, otherwise it is 0. Thus

$$S_1 \lesssim \|B\|_{L_p^n \longrightarrow L_p^n} \left\{ \sum_{j=1}^{n} |I_{j,n}| \left[ |Pw(x_{jn})| \Psi_R(x_{jn}) \right]^p \right\}^{1/p}$$

and so we must show that independently of $n$,

$$\|B\|_{L_p^n \longrightarrow L_p^n} \lesssim 1.$$

We apply Lemma 3.2 with the discrete measure space $\Omega := \{1, 2, \ldots n\}$ and $\mu(\{j\}) = 1$, $j = 1, 2, \ldots n$. Moreover, we set there,

$$k(k,j) := b_{k,j};\ r_{k,j} := \left( \frac{|I_{j,n}|}{|I_{k,n}|} \frac{1 + |x_{kn}|}{1 + |x_{jn}|} \right)^{1/pq}.$$

Thus it suffices to show that

$$\sup_{\{k \mid |x_{kn}| \geq \frac{a_n}{2}\}} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{|I_{j,n}|^2}{(x_{j,n} - x_{k,n})^2} \left( \frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})} \right)^3 \left( \frac{1 + |x_{kn}|}{1 + |x_{jn}|} \right)^{R + 1/p} \lesssim 1, \tag{3.8}$$

and

$$\sup_{j} \sum_{\substack{|x_{kn}| \geq \frac{a_n}{2} \\ k \neq j}}^{n} \frac{|I_{j,n}| \, |I_{k,n}|}{(x_{j,n} - x_{k,n})^2} \left( \frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})} \right)^3 \left( \frac{1 + |x_{kn}|}{1 + |x_{jn}|} \right)^{1/q - R} \lesssim 1. \tag{3.9}$$

This follows using the method of ([8], pp 750-753), recalling that given fixed $\beta \in (0,1)$, we have uniformly for $n$ and $1 \leq l \leq n$

$$|I_{l,n}| \sim \frac{a_n}{n} \left( 1 - \frac{|x_{l,n}|}{a_n} + (nT(a_n))^{-2/3} \right)^{1/2} \sim \frac{a_n}{n} \zeta_n^2(x_{ln}),\ |x_{l,n}| \leq a_{\beta n}$$

and

$$|I_{l,n}| \sim \frac{a_n}{n} T(a_n)^{-1} \left( 1 - \frac{|x_{l,n}|}{a_n} + (nT(a_n))^{-2/3} \right)^{-1/2} \sim \frac{a_n}{n} T(a_n)^{-1} \zeta_n^{-2}(x_{ln}),\ |x_{l,n}| \geq a_{\beta n}.$$

Now we will prove (3.8) and (3.9). For notational simplicity, we will often write just $\Sigma$ together with the indices of summation instead of the full sum. Firstly for (3.8), we may write

$$\sup_{\{k||x_{kn}|\geq \frac{a_n}{2}\}} \sum_{\substack{j=1 \\ j\neq k}}^{n} \leq \sup_{\{k||x_{kn}|\geq \frac{a_n}{2}\}} \sum_{|x_{jn}|\leq \frac{a_n}{3}} + \sup_{\{k||x_{kn}|\geq \frac{a_n}{2}\}} \sum_{\substack{|x_{jn}|\geq \frac{a_n}{3} \\ j\neq k}} .$$

For $|x_{kn}| \geq a_n/2$ and $|x_{jn}| \leq a_n/3$, since $|x_{kn} - x_{jn}| \sim a_n$, $\zeta_n(x_{kn}) \lesssim 1$ and $\zeta_n(x_{jn}) \sim 1$, we have

$$\sum_{|x_{jn}|\leq \frac{a_n}{3}} \lesssim \frac{a_n^{R+1/p}}{a_n^2} \sum_{|x_{jn}|\leq \frac{a_n}{3}} |I_{jn}|^2 \lesssim \frac{a_n^{R+1/p}}{n} \lesssim 1.$$

On the other hand, for $|x_{kn}| \geq a_n/2$,

$$\sum_{\substack{|x_{jn}|\geq a_n/3 \\ j\neq k}} \lesssim \sum_{\substack{a_n/3\leq |x_{jn}|\leq a_{n/3} \\ j\neq k}} + \sum_{\substack{a_{n/3}\leq |x_{jn}|\leq (1+L\delta_n)a_n \\ j\neq k}} \frac{|I_{j,n}|^2}{(x_{j,n}-x_{k,n})^2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 .$$

Thus for the first term in the above, we have

$$\sum_{\substack{a_n/3\leq |x_{jn}|\leq a_{n/3} \\ j\neq k}} \lesssim \frac{a_n}{n}\zeta_n^3(x_{kn}) \sum_{\substack{a_n/3\leq |x_{jn}|\leq a_{n/3} \\ j\neq k}} \frac{|I_{j,n}|}{(x_{j,n}-x_{k,n})^2} \frac{1}{\zeta_n(x_{jn})}$$

and then for $a_n/2 \leq |x_{kn}| \leq a_{n/2}$

$$\begin{aligned}
&\sum_{\substack{a_n/3\leq |x_{jn}|\leq a_{n/3} \\ j\neq k}} \frac{|I_{j,n}|}{(x_{j,n}-x_{k,n})^2} \frac{1}{\zeta_n(x_{jn})} \\
\lesssim\ & \int_{\substack{a_n/3\leq |t|\leq a_{n/3} \\ |t-x_{kn}|\geq C\frac{a_n}{n}\zeta_n^2(x_{kn})}} \frac{\zeta_n^{-1}(t)}{(t-x_{kn})^2}dt \\
\lesssim\ & \int_{a_n/3}^{x_{kn*}} + \int_{x_{kn}^*}^{\frac{a_n+x_{kn}^*}{2}} + \int_{\frac{a_n+x_{kn}^*}{2}}^{a_n} \frac{\zeta_n^{-1}(t)}{(t-x_{kn})^2}dt \\
\lesssim\ & \frac{n}{a_n}\zeta_n^{-3}(x_{kn}) + a_n^{-1}\zeta_n^{-5}(x_{kn})
\end{aligned}$$

where $x_{kn*} := x_{kn} - C\frac{a_n}{n}\zeta_n^2(x_{kn})$ and $x_{kn}{}^* := x_{kn} + C\frac{a_n}{n}\zeta_n^2(x_{kn})$. So we have

$$\begin{aligned}
&\frac{a_n}{n}\zeta_n^3(x_{kn}) \sum_{\substack{a_n/3\leq |x_{jn}|\leq a_{n/3} \\ j\neq k}} \frac{|I_{j,n}|}{(x_{j,n}-x_{k,n})^2} \frac{1}{\zeta_n(x_{jn})} \\
\lesssim\ & 1 + \frac{\zeta_n^{-2}(x_{kn})}{n} \lesssim 1.
\end{aligned}$$

At the same time, for $|x_{kn}| \geq a_{n/2}$, we may easily show that

$$\frac{a_n}{n}\zeta_n^3(x_{kn}) \sum_{\substack{a_{n/3} \leq |x_{jn}| \leq a_{n/3} \\ j \neq k}} \frac{|I_{j,n}|}{(x_{j,n} - x_{k,n})^2} \frac{1}{\zeta_n(x_{jn})} \lesssim 1.$$

Now, we estimate the second term:

Firstly observe that

$$\begin{aligned}
& \sum_{\substack{a_{n/3} \leq |x_{jn}| \leq (1+L\delta_n)a_n \\ j \neq k}} \frac{|I_{j,n}|^2}{(x_{j,n} - x_{k,n})^2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \\
\lesssim \quad & \frac{a_n}{n}\frac{1}{T(a_n)}\zeta_n^3(x_{kn}) \sum_{\substack{a_{n/3} \leq |x_{jn}| \leq (1+L\delta_n)a_n \\ j \neq k}} \frac{|I_{j,n}|\,\zeta_n^{-5}(x_{jn})}{(x_{j,n} - x_{k,n})^2}.
\end{aligned}$$

Then for $a_n/2 \leq |x_{kn}| \leq a_{n/4}$,

$$\begin{aligned}
& \frac{a_n}{n}\frac{1}{T(a_n)}\zeta_n^3(x_{kn}) \sum_{\substack{a_{n/3} \leq |x_{jn}| \leq (1+L\delta_n)a_n \\ j \neq k}} \frac{|I_{j,n}|\,\zeta_n^{-5}(x_{jn})}{(x_{j,n} - x_{k,n})^2} \\
\lesssim \quad & \zeta_n^3(x_{kn})\frac{T(a_n)a_n^{-1}}{n} \sum_{\substack{a_{n/3} \leq |x_{jn}| \leq (1+L\delta_n)a_n \\ j \neq k}} |I_{j,n}|\,\zeta_n^{-5}(x_{jn}) \\
\lesssim \quad & \frac{T(a_n)a_n^{-1}}{n}\delta_n^{-5/4} \lesssim 1
\end{aligned}$$

and for $a_{n/4} \leq |x_{kn}| \leq a_n(1 + L\delta_n)$

$$\begin{aligned}
& \frac{a_n}{n}\frac{1}{T(a_n)}\zeta_n^3(x_{kn}) \sum_{\substack{a_{n/3} \leq |x_{jn}| \leq (1+L\delta_n)a_n \\ j \neq k}} \frac{|I_{j,n}|\,\zeta_n^{-5}(x_{jn})}{(x_{j,n} - x_{k,n})^2} \\
\lesssim \quad & \frac{a_n}{n}\frac{1}{T(a_n)}\zeta_n^3(x_{kn}) \int_{\substack{a_{n/3} \leq |t| \leq a_n(1+L\delta_n) \\ |t - x_{kn}| \geq \frac{a_n}{n}\frac{\zeta_n^{-2}(x_{kn})}{T(a_n)}}} \frac{\zeta_n^{-5}(t)}{(t - x_{kn})^2}dt \\
\lesssim \quad & \frac{a_n}{n}\frac{1}{T(a_n)}\zeta_n^3(x_{kn}) \int_{a_{n/3}}^{x_{kn*}} + \int_{x_{kn}^*}^{\frac{a_n(1+L\delta_n)+x_{kn}^*}{2}} + \int_{\frac{a_n(1+L\delta_n)+x_{kn}^*}{2}}^{a_n(1+L\delta_n)} \frac{\zeta_n^{-5}(t)}{(t - x_{kn})^2}dt \\
\lesssim \quad & 1
\end{aligned}$$

where $x_{kn*}$ and $x_{kn}{}^*$ are as above. Thus (3.8) is proved completely. Now, we prove (3.9). Firstly

$$\sup_j \sum_{\substack{|x_{kn}|\geq \frac{a_n}{2}\\ k\neq j}}^{n} \frac{|I_{j,n}|\,|I_{k,n}|}{(x_{j,n}-x_{k,n})^2}\left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3\left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{1/q-R}$$

$$\lesssim \sup_{|x_{jn}|\leq a_n/3} \sum_{\substack{|x_{kn}|\geq \frac{a_n}{2}\\ k\neq j}}^{n} + \sup_{|x_{jn}|\geq a_n/3} \sum_{\substack{|x_{kn}|\geq \frac{a_n}{2}\\ k\neq j}}^{n}.$$

Since for $|x_{jn}|\leq a_n/3$

$$\sum_{\substack{|x_{kn}|\geq \frac{a_n}{2}\\ k\neq j}}^{n} \lesssim \frac{a_n}{n}\frac{1}{a_n^2}\sum_{|x_{kn}|\geq \frac{a_n}{2}} a_n^{1/q-R} \lesssim \frac{a_n^{1/q-R}}{n} \lesssim 1,$$

we have

$$\sup_{|x_{jn}|\leq a_n/3} \sum_{\substack{|x_{kn}|\geq \frac{a_n}{2}\\ k\neq j}}^{n} \lesssim 1.$$

For the second term, since $|x_{jn}|\geq a_n/3$

$$\sum_{\substack{|x_{kn}|\geq \frac{a_n}{2}\\ k\neq j}}^{n} \lesssim \sum_{\substack{k=1\\ k\neq j}}^{n} \frac{|I_{j,n}|\,|I_{k,n}|}{(x_{j,n}-x_{k,n})^2}\left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3.$$

Thus by [15, Step 4], we have

$$\sum_{\substack{k=1\\ k\neq j}}^{n} \frac{|I_{j,n}|\,|I_{k,n}|}{(x_{j,n}-x_{k,n})^2}\left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \lesssim 1.$$

Therefore, (3.9) is also completely proved. Thus

$$S_1 \lesssim \left\{\sum_{j=1}^{n} |I_{j,n}|\left[|Pw(x_{jn})|\Psi_R(x_{jn})\right]^p\right\}^{1/p}. \tag{3.10}$$

Similarly it follows that

$$S_2 \lesssim \left\{\sum_{j=1}^{n} |I_{j,n}|\left[|Pw(x_{jn})|\Psi_R(x_{jn})\right]^p\right\}^{1/p}. \tag{3.11}$$

Our estimates (3.6), (3.7), (3.10) and (3.11) then show that

$$\begin{aligned} & ||A_1(x)||_{L_p(a_n/2\le|x|\le 2a_n)} \\ \lesssim\ & \left\{\sum_{j=1}^{n}\lambda_{jn}\,|Pw(x_{jn})|^p\,w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp}\right\}^{1/p}. \end{aligned} \tag{3.12}$$

**Step 2-B:** Now we suppose that $|x|\le a_n/2$. We write

$$\begin{aligned} A_1(x) &= a_n^{1/2}\tilde{p}_n w(x)(1+|x|)^R\sum_{j=1}^{n} y_{jn}\frac{\Psi_{-R}(x_{jn})}{(x-x_{jn})} \\ &= a_n^{1/2}\tilde{p}_n w(x)(1+|x|)^R \sum_{|x_{jn}|\ge 2a_n/3} + \sum_{|x_{jn}|\le 2a_n/3} y_{jn}\frac{\Psi_{-R}(x_{jn})}{(x-x_{jn})} \\ &:= A_{11}(x)+A_{12}(x). \end{aligned}$$

Let us estimate both $A_{11}$ and $A_{12}$.

**Step 3-B:** Estimate $||A_{11}(x)||_{L_p(|x|\le a_n/2)}$. Let us write for the given range of $x$,

$$\begin{aligned} & |A_{11}(x)| \\ \sim\ & a_n^2\,(1+|x|)^R\sum_{|x_{jn}|\ge 2a_n/3}\frac{a_n^{-1/2}\,|Pw(x_{jn})|}{a_n^2\,|p_n'w(x_{jn})|\,||y_0|-|x_{jn}||} \\ \sim\ & (1+|x|)^R\sum_{|x_{jn}|\ge 2a_n/3}|Pw(x_{jn})|\,|I_{jn}|(1-|x_{jn}|/a_n+L\delta_n)^{-3/4} \\ \lesssim\ & (1+|x|)^R\sup_{2a_n/3\le|x_{jn}|}|Pw(x_{jn})|\,. \end{aligned}$$

Thus,

$$||A_{11}(x)||_{L_p(|x|\le a_n/2)}\lesssim \sup_{2a_n/3\le|x_{jn}|}|Pw(x_{jn})|\,\left\|(1+|x|)^R\right\|_{L_p(|x|\le a_n/2)}. \tag{3.13}$$

Next we consider:

**Step 4-B:** To estimate $||A_{12}(x)||_{L_p(|x|\le a_n/2)}$, express $A_{12}$ as a sum of two terms. Let us write

$$\begin{aligned} A_{12}(x) &= a_n^{1/2}\tilde{p}_n(x)w(x)(1+|x|)^R\sum_{|x_{jn}|\le 2a_n/3} y_{jn}\left\{\frac{\Psi_{-R}(x_{jn})}{x-x_{jn}}-\frac{1}{|I_{jn}|}H[\chi_{jn}\Psi_{-R}](x)\right\} \\ &\quad +a_n^{1/2}\tilde{p}_n(x)w(x)(1+|x|)^R H\left[\sum_{|x_{jn}|\le 2a_n/3}\frac{y_{jn}}{|I_{jn}|}\chi_{jn}\Psi_{-R}\right](x) \\ &:= J_3(x)+J_4(x). \end{aligned}$$

**Step 5-B:** Estimate $||J_3(x)||_{L_p(|x|\le a_n/2)}$.

For $J_3(x)$ we use the same methods as $J_1(x)$ with

$$\begin{aligned} b_{k,j} &:= \chi_{|x_{kn}|\le a_n/2} \left|I_{j,n}\right|^{2-1/p} \left|I_{k,n}\right|^{1/p} (x_{j,n}-x_{k,n})^{-2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \\ &\quad \times \Psi_{-R}(x_{jn})\Psi_R(x_{kn}) \qquad \text{for } j\neq k. \end{aligned}$$

Thus it suffices to show that

$$\sup_{\{k||x_{kn}|\le \frac{a_n}{2}\}} \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\left|I_{j,n}\right|^2}{(x_{j,n}-x_{k,n})^2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{R+1/p} \lesssim 1, \quad (3.14)$$

and

$$\sup_{j} \sum_{\substack{|x_{kn}|\le \frac{a_n}{2}\\ k\neq j}}^{n} \frac{\left|I_{j,n}\right|\left|I_{k,n}\right|}{(x_{j,n}-x_{k,n})^2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{1/q-R} \lesssim 1. \qquad (3.15)$$

To proceed, we see that in the summations of (3.14) and (3.15), it is enough to show that for $|x_{kn}|\le a_n/2$

$$\sum_{\substack{|x_{jn}|\le \frac{2a_n}{3}\\ j\neq k}} \frac{\left|I_{j,n}\right|^2}{(x_{j,n}-x_{k,n})^2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{R+1/p} \lesssim 1,$$

and for $|x_{jn}|\le 2a_n/3$

$$\sum_{\substack{|x_{kn}|\le \frac{a_n}{2}\\ k\neq j}}^{n} \frac{\left|I_{j,n}\right|\left|I_{k,n}\right|}{(x_{j,n}-x_{k,n})^2} \left(\frac{\zeta_n(x_{kn})}{\zeta_n(x_{jn})}\right)^3 \left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{1/q-R} \lesssim 1.$$

Then since for these ranges, $\zeta_n(x_{kn}) \lesssim 1$ and $\zeta_n(x_{jn}) \lesssim 1$, it is enough to show that for $|x_{kn}|\le a_n/2$

$$\sum_{\substack{|x_{jn}|\le \frac{2a_n}{3}\\ j\neq k}} \frac{\left|I_{j,n}\right|^2}{(x_{j,n}-x_{k,n})^2} \left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{R+1/p} \lesssim 1,$$

and for $|x_{jn}|\le 2a_n/3$

$$\sum_{\substack{|x_{kn}|\le \frac{a_n}{2}\\ k\neq j}}^{n} \frac{\left|I_{j,n}\right|\left|I_{k,n}\right|}{(x_{j,n}-x_{k,n})^2} \left(\frac{1+|x_{kn}|}{1+|x_{jn}|}\right)^{1/q-R} \lesssim 1.$$

The above estimates then easily follow as before using the method of [7, (6.12) and (6.13)].

Therefore, we have shown that

$$\begin{aligned} & ||J_3(x)||_{L_p(|x|\leq a_n/2)} \\ \lesssim \quad & \left\{ \sum_{j=1}^{n} \lambda_{jn} \left| Pw(x_{jn}) \right|^p w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp} \right\}^{1/p} . \end{aligned} \tag{3.16}$$

**Step 6-B:** Estimate $||J_4(x)||_{L_p(|x|\leq a_n/2)}$.

Finally, we estimate $J_4$. Much as before, we deduce that

$$\|H[f]\Psi_r\|_{L_p(\mathbb{R})} \lesssim \|f\Psi_R\|_{L_p(\mathbb{R})}$$

and so

$$\begin{aligned} & \|J_4(x)\|_{L_p(|x|\leq a_n/2)} \\ \lesssim \quad & a_n^2 \left\| |1-|x|/a_n|^{3/4} \Psi_R(x) H\left[ \sum_{|x_{jn}|\leq 2a_n/3} \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \Psi_{-R} \right](x) \right\|_{L_p(|x|\leq a_n/2)} \\ \lesssim \quad & a_n^2 \left\| \Psi_R(x) H\left[ \sum_{|x_{jn}|\leq 2a_n/3} \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \Psi_{-R} \right](x) \right\|_{L_p[-2a_n,2a_n]} \\ \lesssim \quad & a_n^2 \left\| \Psi_R(x) \sum_{|x_{jn}|\leq 2a_n/3} \frac{y_{jn}}{|I_{jn}|} \chi_{jn} \Psi_{-R}(x) \right\|_{L_p[-2a_n,2a_n]} . \end{aligned}$$

Since $|x_{jn}| \leq 2a_n/3$, we have using (3.5) and [7, (2.4)] below

$$|y_{jn}| \sim |Pw(x_{jn})\Psi_R(x_{jn})| \frac{|I_{jn}|}{a_n^2}.$$

Thus

$$\begin{aligned} \|J_4(x)\|_{L_p(|x|\leq a_n/2)} & \lesssim \left\{ \sum_{|x_{jn}|\leq 2a_n/3} |Pw(x_{jn})\Psi_R(x_{jn})|^p \, |I_{jn}| \right\}^{1/p} \\ & \lesssim \left\{ \sum_{j=1}^{n} |Pw(x_{jn})\Psi_R(x_{jn})|^p \, |I_{jn}| \right\}^{1/p} . \end{aligned} \tag{3.17}$$

(3.13), (3.16) and (3.17) then show that

$$\begin{aligned}
& \|A_1(x)\|_{L_p(|x|\leq a_n/2)} \qquad (3.18)\\
\lesssim\ & \left\{\sum_{j=1}^{n} \lambda_{jn} \left|Pw(x_{jn})\right|^p w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp}\right\}^{1/p} \\
+\ & \sup_{2a_n/3\leq|x_{jn}|} |Pw(x_{jn})| \left\|(1+|x|)^R\right\|_{L_p(|x|\leq a_n/2)} \\
\lesssim\ & \left\{\sum_{j=1}^{n} \lambda_{jn} \left|Pw(x_{jn})\right|^p w^{-2}(x_{jn})(1+|x_{jn}|)^{Rp}\right\}^{1/p} + B
\end{aligned}$$

as required. It remains to combine (3.18) with (3.12) and (3.4) to deduce (2.3). □

Next we present:

**The Proof of Theorem 2.1** This follows using the same method as [14, Theorem 1.1]. □

We now consider the proofs of Corollary 2.3 and 2.4. We shall prove Corollary 2.4. The proof of Corollary 2.3 is similar and easier.

**The Proof of Corollary 2.4** Firstly we show that it is enough to assume that $p > 4$. Indeed, assume that Corollary 2.4 holds for $p > 4$. Then for $0 < p \leq 4$, let us choose some constant $q > 4$ satisfying $pq > 4$. Since $\Delta + \hat{\alpha} > 1/p$, we may choose $\Delta_1$ satisfying

$$\Delta - 1/p + 1/pq > \Delta_1 > -\hat{\alpha} + 1/pq$$

and let $q'$ be a conjugate of $q$ (that is, $1/q + 1/q' = 1$). Then by Hölder's inequality

$$\begin{aligned}
& \left\|(L_{n+2}(f,V_{n+2}) - f)(x)w(x)(1+|x|)^{-\Delta}\right\|^p_{L_p(\mathbb{R})} \\
\lesssim\ & \int (L_{n+2}(f,V_{n+2}) - f)^p (x) w^p(x)(1+|x|)^{-\Delta p} dx \\
\lesssim\ & \left(\int (L_{n+2}(f,V_{n+2}) - f)^{pq} (x) w^{pq}(x)(1+|x|)^{-\Delta_1 pq} dx\right)^{1/q} \\
& \times \left(\int (1+|x|)^{-(\Delta-\Delta_1)pq'} dx\right)^{1/q'}.
\end{aligned}$$

Since $\Delta_1 + \hat{\alpha} > 1/pq$, by the result for $p > 4$ we have

$$\lim_{n\to\infty} \left(\int (L_{n+2}(f,V_{n+2}) - f)^{pq} (x) w^{pq}(x)(1+|x|)^{-\Delta_1 pq} dx\right)^{1/q} = 0$$

and since $(\Delta - \Delta_1)pq' > 1$, we have

$$\left( \int (1 + |x|)^{-(\Delta - \Delta_1)pq'} dx \right)^{1/q'} < \infty.$$

Therefore, we have the result for $0 < p \leq 4$.

Thus without loss of generality, assume that $p > 4$. We let $\delta > 0$ and choose a polynomial $P$ so that

$$\|(f - P)(x)w(x)(1 + |x|)^{\alpha}\|_{L_\infty(\mathbb{R})} < \delta.$$

Let $s$ be as in the definition of $B$ in (2.4). Then using (2.3), [7, (2.1) and (2.3)] (4.11) and the fact that for large enough $n$

$$(1 + |y_0|)^{-\alpha} \lesssim (1 + |x|)^{-\alpha}, \; |x| \leq a_{sn}$$

gives for suitable $R$ satisfying $\Delta \geq -R$ and $R > -1/p$

$$\begin{aligned} &\limsup_{n\to\infty} \left\| \left(L_{n+2}(f, V_{n+2}) - f\right)(x) w(x)(1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} \\ \lesssim \;& \delta \|(1 + |x|)^{-\alpha-\Delta}\|_{L_p(\mathbb{R})} + \delta \|(1 + |x|)^{R-\alpha}\|_{L_p(\mathbb{R})}. \end{aligned}$$

Here the constants in $\lesssim$ depend on $\alpha$, $\Delta$ and $p$ but are independent of $\delta$. We now choose

$$0 < \varepsilon < \min\{\alpha, \; \alpha + \Delta - 1/p\}$$

and put

$$R := \alpha - 1/p - \varepsilon.$$

Then $\Delta \geq -R$ and $R > -1/p$. Moreover, $(R - \alpha)p < -1$ and so

$$\limsup_{n\to\infty} \left\| \left(L_{n+2}(f, V_{n+2}) - f\right)(x) w(x)(1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} \lesssim \delta \tag{3.19}$$

where the constants in $\lesssim$ depend on $\alpha$, $\Delta$ and $p$ but are independent of $\delta$. Letting $\delta \to 0^+$ in the above gives the sufficiency of the result. The necessity of Corollary 2.4 follows using the method of [17, Theorem 1.4] and [6, Theorem 1.1]. □

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[1]S. B. Damelin
Department of Mathematics and Computer Science
Georgia Southern University
Post Office Box 809, Statesboro
GA 30460, U.S.A
damelin@gsu.cs.gasou.edu

[2]H. S. Jung and K. H. Kwon
Department of Mathematics
KAIST
Taejon 305-701, KOREA
khkwon@jacobi.kaist.ac.kr
hsjung@math.kaist.ac.kr