Pointwise convergence of derivatives of Lagrange interpolation polynomials for exponential weights

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Abstract

For a general class of exponential weights on the line and on $(-1,1)$, we study pointwise convergence of the derivatives of Lagrange interpolation. Our weights include even weights of smooth polynomial decay near $\pm\infty$ (Freud weights), even weights of faster than smooth polynomial decay near $\pm\infty$ (Erdős weights) and even weights which vanish strongly near ± 1, for example Pollaczek type weights.

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1 Introduction

Let $(-a,a)$ denote the real line $(-\infty,\infty)$ or $(-1,1)$, let $f:(-a,a)\to\mathbb{R}$ be a real valued function and

$$\chi_n := \{x_{1,n}, x_{2,n}, ..., x_{n,n}\}\,,\ n\geq 1$$

a set of pairwise distinct nodes in $(-a,a)$. The *Lagrange interpolation* polynomial to f with respect to χ_n, denoted by $L_n[f,\chi_n] := L_n[f]$, is the unique polynomial of degree at most $n-1$ satisfying

$$L_n[f](x_{j,n}) = f(x_{j,n}),\ 1\leq j\leq n.$$

In this paper, we are interested in studying error estimates for pointwise convergence of $L_n^{(j)}[f]$ to $f^{(j)}$ for fixed $j\geq 0$ whenever f is sufficiently smooth. We will choose χ_n to be a system of n zeroes of a sequence of orthonormal polynomials with respect to a general class of exponential weights on $(-a,a)$ which will be defined more precisely in Section 2 below. Although of independent interest in approximation theory and numerical analysis, such estimates are important, and arise naturally for example in the study of the stability of numerical solutions of various important classes of singular integral equations on $(-a,a)$, which in turn arise in mathematical models dealing with subjects as diverse as hysteretic damping and earthquake shocks. We refer the reader to [5], [6] and the references cited therein for a comprehensive account of some of these vast and interesting applications.

The study of pointwise estimates of Lagrange interpolation for exponential type weights (the case $j = 0$), has been studied extensively by several authors in recent years and there are many good papers on this subject. We refer the reader to the recent papers [5], [12] and the references cited therein for a detailed account of this work. The case $j \geq 1$ is far less studied in the literature and only partial results can be found in works of Balázs and Kanjin and Sakai for even Hermite type weights on $(-\infty, \infty)$, see [1, 11] and Remark 2.6 below. The main idea of this paper is to establish Jackson-Favard type theorems for a general class of exponential weights on $(-a, a)$ with various rates of smooth decay near $\pm a$, [See Example 2.1 and Definition 2.2 below], extending earlier work of Mhaskar and then to combine these tools with results on orthogonal expansions and recent Jackson and Converse Theorems of weighted polynomial approximation which were recently proved by Ditzian-Lubinsky, Damelin-Lubinsky, Damelin and Lubinsky, see [4, 7, 8, 14]. These tools combined, allow us to prove our main result.

We mention that besides the ideas above, another interesting and new feature of this work is to be able formulate a pointwise convergence result, which works for every fixed $j \geq 0$ and simultaneously for even weights of various rates of smooth decay near ± 1 or $\pm\infty$. The smoothness assumptions on our weights are consistent with recent work of Kubayi, see [12].

We now proceed with the statement of our main result which is contained in Section 2 and then to our proofs which are contained in Section 3.

2 Main Result

In this section, we state our main result, Theorem 2.3. To this end, we will need to first introduce some definitions, notation and useful facts.

2.1 Class of Weights and Interpolation Array

In this subsection, we define our class of weights and our interpolation array χ_n.

Throughout, for any two sequences $\{b_n\}_n$ and $\{c_n\}_n$ of nonzero real numbers we shall write $b_n \lesssim c_n$, if there exists a constant $C > 0$, independent of n such that $b_n \leq Cc_n$ for n sufficiently large and we shall write similarly, $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. Similar notation will be used for functions and sequences of functions. We denote by Π_n, the space of polynomials of degree at most n, thus $L_n \in \Pi_{n-1}$.

In order to define our interpolation array, we need a class of exponential weights w on $(-a, a)$ for which the following are archetypal examples:

Example 2.1 • *Even weights on the line of smooth polynomial decay:*

$$w_\alpha := \exp(-Q_\alpha)$$

where

$$Q_\alpha(x) := |x|^\alpha,\ \alpha > 1, \qquad x \in (-\infty, \infty);$$

- *Even weights on the line of faster than smooth polynomial decay:*

$$w_{k,\beta} := \exp(-Q_{k,\beta})$$

with

$$Q_{k,\beta}(x) := \exp_k\left(x^\beta\right) - \exp_k(0),\ x \in (-\infty, \infty),\ \beta > 0,\ k \geq 1.$$

- *Even weights on* $(-1, 1)$ *with fast exponential rates of decay near* ± 1*:*

$$w^{k,\gamma} := \exp(-Q_{k,\gamma})$$

with

$$Q_{k,\gamma}(x) := \exp_k(1 - x^2)^{-\gamma} - \exp_l(1),\ x \in (-1, 1),\ \gamma > 0,\ k \geq 1.$$

Here and throughout, $\exp_k$ *and* $\log_k$ *denote respectively* k*th iterated exponentials and logarithms.*

The weights w_α are called Freud weights in the literature (the Hermite weight is just w_2) and $w_{k,\beta}$ and $w^{k,\gamma}$ are called Erdős and generalised Pollaczek weights respectively. The later are characterised by the fact that they decay much faster than classical Jacobi weights near the endpoints ± 1. See [13] and the references cited therein. The aforementioned examples above are special cases of a general class of *admissible* weights which we now introduce:

Definition 2.2 Class of Admissible Weights *A weight* $w : (-a, a) \to (0, \infty)$ *will be said to be admissible if it satisfies the following conditions below:*

- $Q := \log(1/w)$ *is continuously differentiable, even and satisfies* $Q(0) = 0$*;*
- Q' *is nondecreasing in* $(0, a)$ *with*

$$\lim_{x \to a^-} Q(x) = \infty;$$

- *The function*

$$T(x) := \frac{xQ'(x)}{Q(x)},\ x \neq 0$$

is quasi-increasing in $(0, a)$ *(ie* $T(x) \leq CT(y)$, $0 < x \leq y < a$*) with*

$$T(x) \geq \lambda > 1,\ x \in (0, a);$$

- *There exist positive constants C and C_1 such that*

$$\frac{yQ'(y)}{xQ'(x)} \leq C\left(\frac{Q(y)}{Q(x)}\right)^{C_1}, \quad y \geq x > 0.$$

- *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in (-a, a)\backslash\{0\}$,*

$$\int_{x-\frac{\delta|x|}{T(x)}}^{x+\frac{\delta|x|}{T(x)}} \frac{|Q'(s) - Q'(x)|}{|s-x|^{3/2}} ds \leq \varepsilon |Q'(x)| \sqrt{\frac{T(x)}{|x|}}.$$

- *For every $x \in (0, a)$, we have*

$$Q'(x) w^{-1}(x) \int_x^a w(u) du \lesssim 1.$$

We refer the interested reader to Example 2.1 above, for examples of admissible weights as well as to [5], [13] and the references cited therein for further perspectives and applications. We note, that the function T controls the decay of the weight near $\pm a$, for example in the case of Freud weights, it is uniformly bounded but grows in the case of Erdős or Pollaczek type weights. Observe that Q'' need not exist in the definition above, instead we require only a local Lipschitz condition of Q'. We finally mention that the last condition on w is easily proven if for example

$$\lim_{|x| \to a} \frac{Q''(x)}{Q'(x)^2} = 0$$

which is true for all our prime examples and even more generally, see [13].

Interpolation Array Given an admissible weight w, we let $p_n(w^2; \cdot)$ denote the unique nth degree orthonormal polynomial with respect to w^2

$$p_n(w^2, x) = \gamma_n(w^2) x^n + \textit{lower degree terms } (\gamma_n(d\alpha) > 0);$$

defined by

$$\int_{-a}^{a} p_n(w^2; x) p_m(w^2; x) w^2(x) dx = \delta_{mn},\ m, n = 0, 1, 2...$$

Then χ_n will consist of the n zeroes $\{x_{j,n}\}$, $1 \leq j \leq n$ of $p_n(w^2; \cdot)$ which are contained in $(-a, a)$ and may be ordered as

$$x_{n,n} < x_{n-1,n} < ... < x_{2,n} < x_{1,n}.$$

It follows that

$$L_n[f](x) = \sum_{j=1}^{n} l_{j,n}(w^2; x)$$

where

$$l_{j,n}(w^2;x) := \frac{p_n(w^2;x)}{p_n'(w^2;x_{j,n})(x-x_{j,n})},\ 1 \le j \le n,\ x \in (-a,a).$$

In order to state our main result, we need a damping function which plays the role of $\sqrt{1-x^2}$ in Chebyshev approximation on $(-1,1)$. To this end, and in what follows, we let w be admissible and let a_n, $n \ge 1$ denote the unique positive solution of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1-x^2}}dx.$$

Then, it is well known, see [13], that a_n exists, is unique and grows with n at a rate governed by the following well known fact: For every polynomial $P_n \in \Pi_n$, $n \ge 1$

$$\|P_n w\|_{L_\infty[-a_n,a_n]} = \|P_n w\|_{L_\infty(-a,a)}.$$

Here and in the sequel, $L_p(-a,a)$ denotes the space of all real valued L_p functions.

The numbers a_n are needed as scaling factors to define the sequence of functions

$$\phi_t := |1-|x|/\sigma(t)|^{1/2} + T^{-1/2}(\sigma(t)), \quad x \in (-a,a) \tag{2.1}$$

where

$$\sigma(t) := \inf\{a_u : a_u/u \le t,\ t > 0\}. \tag{2.2}$$

Finally, we recall that for $0 < p \le \infty$ and $fw \in L_p(-a,a)$,

$$E_n[f]_{w,p} := \inf_{P \in \mathcal{P}_n} \|(f-P)\,w\|_p \tag{2.3}$$

denotes the error of best weighted polynomial approximation to f.

We are ready to state our main result. This is contained in

Theorem 2.3 Main Result *Let w be admissible, $f : (-a,a) \to \mathbb{R}$ and suppose $f^{(j)}$ exists for some $j \ge 0$. Recall the functions $\phi_.$ given by (2.1) and the error of best weighted polynomial approximation given by (2.3). Then uniformly for $n \ge 1$, $x \in (-a,a)$ and f*

$$\left|L_n^{(j)}[f](x) - f^{(j)}(x)\right| \phi^j_{\frac{a_n}{n}}(x)w(x) \lesssim \left(\|L_n\|_\infty + T^{1/4}(a_n)\right) E_{n-j-1}[f^{(j)}]_{w,\infty}$$

where

$$\|L_n\|_\infty := \left\|\sum_{k=1}^{n} \left|l_{kn} w^{-1}(x_{kn}) w\right|\right\|_{L_\infty(-a,a)}.$$

Remark 2.4 It can be shown using the results of [12], that Theorem 2.3 reduces to the classical Lebesgue's inequality in the case $j = 0$ as it should. Moreover, it is well known, see [4, 7, 8, 14], that for admissible w, $\sigma(a_n/n) \sim a_n$ uniformly in n thus away from a_n, $\phi^{-1}_{\frac{a_n}{n}}(\cdot) \sim 1$ and close to a_n, $\phi^{-1}_{\frac{a_n}{n}}(\cdot) \sim T(a_n)^{1/2}$ which is uniformly bounded for Freud weights.

Thus in the case of Freud weights, Theorem 2.3 is Lebesgue like with the growth on the right coming only from the Lebesgue constant. We find it convenient to illustrate this by means of

Corollary 2.5 *Let $w = w_\alpha$, $\alpha > 1$ be given by Example 2.1. Let $j \geq 0$, $k \geq 1$ and suppose that $f^{(j)}$ is continuous with $f^{(j)}w$ vanishing at $\pm\infty$. Suppose also that $f^{(j+k)}w \in L_\infty(\mathbb{R})$. Then*

$$\left|L_n^{(j)}[f](x) - f^{(j)}(x)\right| w(x) \lesssim n^{k(\frac{1}{\alpha}-1)+1/6}.$$

Remark 2.6 As far as we know, the most general results dealing with pointwise convergence of derivatives of Lagrange interpolation for exponential weights, apart from ours, are due to Balázs and Kanjin and Sakai, see [1, 11] who are able to treat Corollary 2.5 only in the case when α is an even integer. [Balázs, $\alpha = 2$; Kanjin and Sakai, α an even integer]. Even in this special case, the main result of Kanjin and Sakai only covers a restricted range of x and has a damping factor which does not take into the account the behaviour of x close to a_n. Indeed in the case of Freud weights and in particular Corollary 2.5, we see that no damping effect occurs near $\pm a_n$. We refer the reader to the paper [11] for further details.

Remark 2.7 In the case when w is admissible and T grows, rates of convergence may also be obtained in Theorem 2.3. In these cases however, we see that endpoint effects near $\pm a_n$ do come into play from the function $\phi_{\cdot}$. In this respect, we find it constructive to state the following 2 further corollaries of Theorem 2.3.

Corollary 2.8 *Let $w = w_{k,\beta}$, $\beta > 0$ be given by Example 2.1. Let $j \geq 0$, $k \geq 1$ and suppose that $f^{(j)}$ is continuous with $f^{(j)}w$ vanishing at $\pm\infty$. Suppose also that $f^{(j+k)}w \in L_\infty(\mathbb{R})$. Then*

$$\left|L_n^{(j)}[f](x) - f^{(j)}(x)\right| w(x) \left[\left|1 - \frac{|x|}{a_n}\right| + T^{-1}(a_n)\right]^{1/2} \lesssim n^{1/6}T^{1/6}(a_n)\left(\frac{a_n}{n}\right)^k$$

where

$$a_n \sim (\log_k n)^{\frac{1}{\beta}}$$

and

$$T(a_n) \sim \prod_{j=1}^{k} \log_j n.$$

Corollary 2.9 *Let $w = w^{l,\gamma}$, $\gamma > 0$ be given by Example 2.1. Let $j \geq 0$, $k \geq 1$ and suppose that $f^{(j)}$ is continuous with $f^{(j)}w$ vanishing at ± 1. Suppose also that $f^{(j+k)}w \in L_\infty(-1,1)$. Then*

$$\left|L_n^{(j)}[f](x) - f^{(j)}(x)\right| w(x) \left[\left|1 - \frac{|x|}{a_n}\right| + T^{-1}(a_n)\right]^{1/2} \lesssim n^{1/6} T^{1/6}(a_n) n^{-k}$$

where

$$a_n \sim \begin{cases} 1 - n^{\frac{1}{\gamma+1/2}}, & l = 0 \\ 1 - (\log_l n)^{-\frac{1}{2\gamma}}, & l \geq 1 \end{cases}$$

and

$$T(a_n) \sim \begin{cases} n^{\frac{1}{\gamma+1/2}}, & l = 0 \\ (\log_l n)^{1+\frac{1}{\gamma}} \prod_{j=1}^{l-1} \log_j n, & l \geq 1. \end{cases}$$

We proceed with our proofs. These are contained in Section 3.

3 Proofs

In this section, we prove our main result, namely Theorem 2.3 and its Corollaries. Throughout this section, w will be henceforth, a fixed admissible weight.

3.1 Jackson and Converse Theorems of Polynomial Approximation

Our first important tool is Jackson and Converse Theorems of Polynomial Approximation.

For $h > 0$, an interval J, $r \geq 1$ and $f : (-a,a) \to \mathbb{R}$, we define

$$\Delta_h^r(f,x,J) := \begin{cases} \sum_{i=0}^r \binom{r}{i} (-1)^i f\left(x + \frac{rh}{2} - ih\right), & x \pm \frac{rh}{2} \in J \\ 0, & \text{otherwise} \end{cases}$$

to be the rth symmetric difference of f. If J is not specified, it will be taken as $(-a,a)$.

We further recall, (see (2.1), (2.2)), the sequence of functions

$$\phi_t := |1 - |x|/\sigma(t)|^{1/2} + T^{-1/2}(\sigma(t)), \quad x \in (-a,a)$$

where

$$\sigma(t) := \inf\{a_u : a_u/u \leq t,\, t > 0\}.$$

Then for $0 < p \leq \infty$ and $r \geq 1$, the weighted modulus of smoothness of f is given by

$$\begin{aligned} \omega_{r,p}(f,w,t) := \quad & \sup_{0<h\leq t} \left\| w\left(\Delta^r_{h\phi_t(x)}(f)\right)\right\|_{L_p(|x|\leq\sigma(2t))} \\ & + \inf_{R\in\Pi_{r-1}} \|(f-R)\,w\|_{L_p(|x|\geq\sigma(4t))}. \end{aligned}$$

The following Jackson-Favard and Converse Theorems follow from the work of Ditzian-Lubinsky, Damelin-Lubinsky, Damelin and Lubinsky, see [4, 7, 8, 14].

Theorem 3.1 Jackson-Favard and Converse Theorems

(a) *Let $0 < p \le \infty$ and $r \ge 1$. Then for all $f : (-a, a) \to \mathbb{R}$ for which $fw \in L_p(-a, a)$ (and for $p = \infty$, we require f to be continuous, and fw to vanish at $\pm a$), we have uniformly for f and $n \ge 1$*

$$E_n[f]_{w,p} \lesssim w_{r,p}\left(f, w, \frac{a_n}{n}\right).$$

(b) *Moreover if $1 \le p \le \infty$ and $f^{(r)}w \in L_p$, we have uniformly for f and small enough $t > 0$*

$$w_{r,p}\left(f, w, t\right) \lesssim t^r \left\|f^{(r)}\phi_t^r w\right\|_p.$$

(c) *Let $1 \le p \le \infty$ and let $f'w \in L_p(-a, a)$ (with f' continuous and $f'w$ vanishing at $\pm\infty$ if $p = \infty$). Then*

$$E_n[f]_{w,p} \lesssim \frac{a_n}{n} E_{n-1}[f']_{w,p}.$$

3.2 Orthogonal expansions

In this subsection, we study orthogonal expansions for admissible weights.

We begin by introducing some auxiliary quantities which we will find useful in the sequel. To this end, set throughout

$$\delta_n := (nT(a_n))^{-2/3}, \quad n \ge 1,$$

$$\Psi_n(x) := \begin{cases} \max\left\{\sqrt{1 - |x|/a_n + L\delta_n}, \frac{1}{T(a_n)\sqrt{1-|x|/a_n+L\delta_n}}\right\}, & |x| \le a_n \\ \Psi_n(a_n), & |x| > a_n. \end{cases} \tag{3.1}$$

For $p \ge 1$ and $f \in L^p_w$, we also define

$$b_k(f) := b_k(w^2; f) := \int f(t)p_k(t)w^2(t)dt, \qquad k = 0, 1, \cdots$$

$$s_m(f, x) := s_m(w^2; f, x) := \sum_{k=0}^{m-1} b_k(f)p_k(t), \qquad m = 1, 2, \cdots,$$

and

$$v_n(f,x) := v_n(w^2;f,x) := \frac{1}{n}\sum_{m=n+1}^{2n} s_m(f,x) \qquad n=1,2,\cdots.$$

We have the following proposition, describing some of properties of the operators v_n.

Proposition 3.2 *Let $n \geq 1$ be an integer, $1 \leq p \leq \infty$, and $p' := p/(p-1)(= \infty \, if \, p=1)$. If $fw \in L_p$ then $v_n(f) \in \Pi_{2n-1}$ and*

$$v_n(P,x) = P(x), \qquad x \in \mathbb{R},\ P \in \Pi_n.$$

Duality Principle : *The operator v_n is self adjoint in the sense that if $fw \in L_p$ and $gw \in L_{p'}$ then*

$$\int f(x)v_n(g,x)w^2(x)dx = \int v_n(f,x)g(x)w^2(x)dx.$$

Proof. The only part of the proposition that requires a proof is the duality principle which is easily verified by a direct calculation. □.

Our remaining plan in this subsection, is to prove the following result dealing with the boundedness of the operators v_n and an important corollary.

Theorem 3.3 Boundedness of v_n : *Let $1 \leq p \leq \infty$, $fw \in L_p$. Then, uniformly for $n \geq 1$,*

$$\|wv_n(f)\|_{L_p} \lesssim T^{1/4}(a_n)\,\|wf\|_{L_p}. \tag{3.2}$$

In particular, when $p = \infty$,

$$|w(x)v_n(f,x)| \lesssim \Psi_n^{-1/2}(x)\,\|wf\|_{\infty} \tag{3.3}$$

where Ψ_n is given by (3.1).

We remark that Theorem 3.3 was first established by Freud for a subclass of Freud admissible weights, see([9, 10, 18]). Various versions of Theorem 3.3 have been proved by Lubinsky, Mache, Mthembu and Mashele, see ([16, 15, 17, 18]) and the references cited therein. The basic method of proof we use goes back to Freud and we choose to provide full details for clarity and the reader's convenience.

Proof of Theorem 3.3. We first consider the case when $p = \infty$. Let $x \in (-a,a)$ be fixed and define

$$f_1(t) := \begin{cases} f(t), & \text{if } |x-t| \leq \frac{a_n}{n} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_2(t) := f(t) - f_1(t).$$

Defining the Christoffel Darboux kernel,

$$K_m(x,t) = \sum_{k=0}^{m-1} p_k(x)p_k(t),$$

we recall that we have that

$$s_m(f,x) = \int f(t)K_m(x,t)w^2(t)dt$$

and

$$\int K_m^2(x,t)w^2(t)dt = K_m(x,x) = \sum_{k=0}^{m-1} p_k^2(x).$$

Now, we apply Schwarz's inequality with the above and obtain for every integer m with $n+1 \leq m \leq 2n$,

$$\begin{aligned}
&w(x)\,|s_m(f_1,x)| \\
&\leq w(x) \int |f_1(t)|\,|K_m(x,t)|\,w^2(t)dt \\
&\leq w(x) \left(\int_{|x-t|\leq a_n/n} |f(t)w(t)|^2\,dt \right)^{1/2} \left(\int K_m^2(x,t)w^2(t)dt \right)^{1/2} \\
&\lesssim \|fw\|_\infty\,\Psi_m^{-1/2}(x)
\end{aligned}$$

so that

$$w(x)\,|v_n(f_1,x)| \lesssim \Psi_n^{-1/2}(x)\,\|fw\|_\infty\,.$$

Let

$$f_2^*(t) := \frac{f_2(t)}{x-t}, \;\; x \neq t.$$

Then we have

$$\begin{aligned}
s_m(f_2,x) &= \frac{\gamma_{m-1}}{\gamma_m} \int f_2^*(t)\left(p_m(t)p_{m-1}(x) - p_{m-1}(t)p_m(x)\right)w^2(t)dt \\
&= \frac{\gamma_{m-1}}{\gamma_m}\left(b_m(f_2^*)p_{m-1}(x) - b_{m-1}(f_2^*)p_m(x)\right)
\end{aligned}$$

where γ_m is the leading coefficient of m-th degree orthonormal polynomial $p_m(w^2,\cdot)$, see p. 4. Therefore,

$$\begin{aligned}
|v_n(f_2,x)| &\leq \frac{1}{n} \sum_{m=n+1}^{2n} |s_m(f_2,x)| \\
&\lesssim \frac{a_n}{n} \left(\sum_{m=n+1}^{2n} \left(|b_m(f_2^*)p_{m-1}(x)| + |b_{m-1}(f_2^*)p_m(x)|\right) \right).
\end{aligned}$$

Here we use that uniformly for $m \geq 1$

$$\frac{\gamma_{m-1}}{\gamma_m} \sim a_m.$$

See [13]. Applying Schwarz's inequality again, we see that

$$\begin{aligned} w(x)\left|v_n(f_2,x)\right| &\lesssim \frac{a_n}{n}\left(w^2(x)\sum_{k=0}^{2n} p_k^2(x)\right)^{1/2}\left(\sum_{k=0}^{2n}\left|b_k(f_2^*)\right|^2\right)^{1/2} \\ &\lesssim \sqrt{\frac{a_n}{n}}\Psi_{2n}^{-1/2}(x)\left(\sum_{k=0}^{2n}\left|b_k(f_2^*)\right|^2\right)^{1/2}. \end{aligned}$$

Now we observe that Bessel's inequality and the orthogonality of p_k imply that

$$\begin{aligned} \|f_2^* w\|_2^2 &= \|w(f_2^* - s_{2n}(f_2^*))\|_2^2 + \|w s_{2n}(f_2^*)\|_2^2 \\ &\geq \|w s_{2n}(f_2^*)\|_2^2 = \sum_{k=0}^{2n}\left|b_k(f_2^*)\right|^2. \end{aligned}$$

Moreover an easy estimate yields

$$\begin{aligned} \|f_2^* w\|_2^2 &= \int\left|\frac{f_2(t)}{x-t}\right|^2 w^2(t)dt \\ &= \int_{|x-t|\geq a_n/n}\frac{|f(t)w(t)|^2}{(x-t)^2}dt \\ &\lesssim \frac{a_n}{n}\|fw\|_\infty. \end{aligned}$$

Thus we learn that

$$w(x)\left|v_n(f_2,x)\right| \lesssim \Psi_n^{-1/2}(x)\|fw\|_\infty.$$

Therefore, we have

$$w(x)\left|v_n(f,x)\right| \lesssim \Psi_n^{-1/2}(x)\|fw\|_\infty \lesssim T^{1/4}(a_n)\|fw\|_\infty.$$

This proves (3.2) and (3.3) for $p=\infty$. To see (3.2) for $p=1$, we use the duality principle, Proposition 3.2 and the case $p=\infty$. Note that for $f \in L_w^1$,

$$\begin{aligned} \|f v_n(f)\|_1 &= \sup_{\|gw\|_\infty=1}\left|\int v_n(f,x)g(x)w^2(x)dx\right| \\ &= \sup_{\|gw\|_\infty=1}\left|\int v_n(g,x)f(x)w^2(x)dx\right| \\ &\leq \|fw\|_1 \sup_{\|gw\|_\infty=1}\|v_n(g)w\|_\infty \\ &\lesssim T^{1/4}(a_n)\|fw\|_1. \end{aligned}$$

The Riesz-Thorin interpolation theorem then yields the general result. □

We deduce

Corollary 3.4 *Let $1 \leq p \leq \infty$, $fw \in L_p(-a,a)$ and $n \geq 1$. Then*

$$E_{2n}[f]_{w,p} \leq \left\| w\left(f - v_n(f)\right)\right\|_p \lesssim T^{1/4}(a_n) E_n[f]_{w,p}.$$

Proof. We need to prove the second inequality since the first is clear. Since we have for any polynomial $P \in \Pi_n$

$$f - v_n(f) = (f-P) - v_n(f-P),$$

if we choose a polynomial $P \in \Pi_n$ satisfying

$$\left\|(f-P)\, w\right\|_p \lesssim E_n[f]_{w,p}$$

then we have from Theorem 3.3

$$\left\|(f - v_n(f))\, w\right\|_p \lesssim T^{1/4}(a_n) E_n[f]_{w,p}$$

as required. □

3.3 Jackson-Favard Result

Our primary aim in this section is to prove the following interesting

Theorem 3.5 Jackson-Favard Result : *Let $1 \leq p \leq \infty$, $r \geq 1$ and $f^{(r)}w \in L_p(-a,a)$. Suppose that for some $n \geq 1$, $P \in \Pi_n$ and $\eta > 0$*

$$\left\|\left(f^{(r-1)} - P^{(r-1)}\right) \phi_{\frac{a_n}{n}}^{r-1} w\right\|_p \leq \eta.$$

Then

$$\left\|\left(f^{(r)} - P^{(r)}\right) \phi_{\frac{a_n}{n}}^{r} w\right\|_p \lesssim \left\{T^{1/4}(a_n) E_{n-r}[f^{(r)}]_{w,p} + \frac{n}{a_n}\eta\right\}.$$

In particular, if

$$\left\|\left(f^{(r-1)} - P^{(r-1)}\right) \phi_{\frac{a_n}{n}}^{r-1} w\right\|_p \lesssim T^{1/4}(a_n) E_{n-r+1}[f^{(r-1)}]_{w,p},$$

then we have

$$\left\|\left(f^{(r)} - P^{(r)}\right) \phi_{\frac{a_n}{n}}^{r} w\right\|_p \lesssim T^{1/4}(a_n) E_{n-r}[f^{(r)}]_{w,p}.$$

The remainder of this subsection is devoted to the proof of Theorem 3.5.

We begin with the following Lemma which is a generalization of [17, Lemma 4.1.4].

Lemma 3.6 *Define*

$$I(h)(t) := w^{-2}(t)\int_t^a w^2(u)h(u)du,\ t\in(-a,a),\ w^2h\in L_1(-a,a).$$

Further let $1\le p\le\infty$, *and* $p' := p/(p-1)(=\infty$ *if* $p=1)$. *Suppose in addition that* $wh\in L_p(-a,a)$, *and*

$$\int_{-a}^a w^2(t)h(t)dt = 0. \tag{3.4}$$

Then

$$\|w[I(h)]'\|_p \lesssim \|wh\|_p. \tag{3.5}$$

Moreover, if g *is absolutely continuous, and* $wg'\in L_{p'}(-a,a)$ *then*

$$\int_{-a}^a g(x)h(x)w^2(x)dx = \int_{-a}^a g'(t)I(h)(t)w^2(t)dt. \tag{3.6}$$

Proof. We observe first that as $w(a)=0$,

$$\begin{aligned}&[I(h)]'(t)w(t) = 2Q'(t)w^{-1}(t)\int_t^a w^2(u)h(u)du - h(t)w(t)\\ &= 2Q'(t)w(t)I(h)(t) - h(t)w(t).\end{aligned}$$

Thus to establish (3.5), we need to show that

$$||Q'wI(h)||_{L_p(-a,a)} \lesssim ||wh||_{L_p(-a,a)}. \tag{3.7}$$

Firstly, because

$$Q'(t)w^{-1}(t)\int_t^a w(u)du \lesssim 1$$

for all positive t, we see that

$$|Q'(t)w(t)I(h)(t)| \lesssim ||wh||_{L_\infty(-a,a)}$$

holds for all positive t. An application of (3.4), the method of [17, Lemma 4.1.4] and the Riesz-Thorin interpolation theorem yields (3.7) for $p=\infty$, $p=1$ and hence all p. Thus (3.5) holds. (3.6) follows exactly as in [17, Lemma 4.1.4]. □

Next we need

Lemma 3.7 *Let* $1\le p\le\infty$, $p' := p/(p-1)(=\infty$ *when* $p=1)$. *Let* f *be absolutely continuous with* $f'w\in L_p(-a,a)$. *Then for* $n\ge1$, *there exists a polynomial* $V_n := V_n(f)\in\Pi_{2n}$ *such that* $V_n' = v_n(f')$ *and*

$$\|w(f-V_n)\|_p \lesssim \frac{a_n}{n}T^{1/4}(a_n)E_n[f']_{w,p}. \tag{3.8}$$

Proof. Without loss of generality, we may assume that $f(0) = 0$. Let

$$G(x) := \int_0^x f'(t) - v_n(f', t)dt,$$

and choose a constant a such that

$$\|w\,(G - a)\|_p \leq 2E_0[G]_{w,p}.$$

Then define

$$V_n(x) := a + \int_0^x v_n(f', t)dt \in \Pi_{2n}.$$

We conclude that

$$\|w\,(f - V_n)\|_p = \|w\,(G - a)\|_p \leq 2E_0[G]_{w,p}.$$

By the duality principle, we may choose h with $hw \in L_{p'}(-a, a)$ such that (3.4) is satisfied, $\|hw\|_{L_{p'}(-a,a)} = 1$ and

$$\|w\,(f - V_n)\|_p \leq 2E_0[G]_{w,p} \leq 4\left|\int G(x)h(x)w^2(x)dx\right|.$$

We now apply (3.4), (3.5), (3.6), and Corollary 3.4 to deduce that

$$\begin{aligned}
&\|w\,(f - V_n)\|_p \\
&\leq 4\left|\int_{-a}^{a} G(x)h(x)w^2(x)dx\right| \\
&= 4\left|\int_{-a}^{a} \left(f'(t) - v_n(f', t)\right) I(h)(t)w^2(t)dt\right| \\
&\leq 4\left\|w\left(f' - v_n(f')\right)\right\|_p E_n[I(h)]_{w,p'} \\
&\lesssim T^{1/4}(a_n)E_n[f']_{w,p}E_n[I(h)]_{w,p'}
\end{aligned}$$

where we use the fact that for any polynomial $\pi_n \in \Pi_n$,

$$\begin{aligned}
&\left|\int_{-a}^{a} \left(f'(t) - v_n(f', t)\right) I(h)(t)w^2(t)dt\right| \\
&= \left|\int_{-a}^{a} \left(f'(t) - v_n(f', t)\right) \left(I(h)(t) - \pi_n(t)\right) w^2(t)dt\right|
\end{aligned}$$

since $\int_{-a}^{a} \left(f'(t) - v_n(f', t)\right) \pi_n(t)w^2(t)dt = 0$. Finally we observe that by Theorem 3.1 and Lemma 3.6 we have

$$E_n[I(h)]_{w,p'} \lesssim \frac{a_n}{n}\left\|w[I(h)]'\phi_{\frac{a_n}{n}}\right\|_{p'} \lesssim \frac{a_n}{n}\left\|w[I(h)]'\right\|_{p'} \lesssim \frac{a_n}{n}\|wh\|_{p'} = \frac{a_n}{n}.$$

This yields the result. $\square$

We are now able to establish a special case of Theorem 3.5 namely

Lemma 3.8 *Let $1 \le p \le \infty$ and f be a function with $f'w \in L_p$. If for $n \ge 1$, $P \in \Pi_n$ and $\eta > 0$ we have*

$$\|w(f-P)\|_p \le \eta,$$

then

$$\left\|(f'-P')\,\phi_{\frac{a_n}{n}} w\right\|_p \lesssim \left\{T^{1/4}(a_n)E_{n-1}[f']_{w,p} + \frac{n}{a_n}\eta\right\}.$$

In particular, if

$$\|w(f-P)\|_p \lesssim T^{1/4}(a_n)E_n[f]_{w,p},$$

then we also have

$$\left\|(f'-P')\,\phi_{\frac{a_n}{n}} w\right\|_p \lesssim T^{1/4}(a_n)E_{n-1}[f']_{w,p}.$$

Proof. We find a polynomial V_{n-1} as in Lemma 3.7, so that $V'_{n-1} = v_{n-1}(f')$ and (3.8) is satisfied with $n-1$ instead of n. Thus we may apply Lemma 3.7 and Markov's Inequality, see [13] to obtain

$$\begin{aligned}\left\|(f'-P')\,\phi_{\frac{a_n}{n}} w\right\|_p \quad &\le \left\|(f'-v_{n-1}(f'))\,\phi_{\frac{a_n}{n}} w\right\|_p + \left\|\left(V'_{n-1}-P'\right)\phi_{\frac{a_n}{n}} w\right\|_p \\ &\lesssim T^{1/4}(a_n)E_{n-1}[f']_{w,p} + \frac{n}{a_n}\left\|(V_{n-1}-P)\,w\right\|_p. \quad (3.9)\end{aligned}$$

To complete the proof, it suffices to apply Lemma 3.7 again with (3.9) to deduce that

$$\begin{aligned}\|w(V_{n-1}-P)\|_p &\le \|w(f-V_{n-1})\|_p + \|w(f-P)\|_p \\ &\lesssim \frac{a_n}{n}T^{1/4}(a_n)E_{n-1}[f']_{w,p} + \eta.\end{aligned}$$

Inserting this last estimate into (3.9) gives the result. □

We are now able to present the

Proof of Theorem 3.5. We proceed as in Lemma 3.8, except that we find a polynomial V_{n-1} as in Lemma 3.7, so that $V'_{n-1} = v_{n-1}(f^{(r)})$ and (3.8) is satisfied with $n-1$ instead of n. The rest of the proof is as in Lemma 3.8. □

3.4 The Proof of Theorem 2.3 and Corollaries 2.5, 2.8, 2.9

In this subsection, we prove Theorem 2.3 and Corollaries 2.5, 2.8, 2.9.

We begin with the

Proof of Theorem 2.3. Let $P_n \in \Pi_n$ be a polynomial of degree at most n satisfying

$$\|(f-P_{n-1})w\|_p \lesssim E_{n-1}[f]_{w,p}.$$

Then by Theorem 3.5, we have

$$\left\|(f^{(j)} - P_{n-1}^{(j)})\phi^j_{\frac{a_n}{n}} w\right\|_p \lesssim T^{1/4}(a_n) E_{n-1-j}[f^{(j)}]_{w,p}.$$

On the other hand, Theorem 3.1 implies that

$$E_{n-1}[f]_{w,\infty} \lesssim \left(\frac{a_n}{n}\right)^j E_{n-1-j}[f^{(j)}]_{w,\infty}.$$

Let

$$\|L_n\|_\infty := \left\|\sum_{k=1}^n |l_{kn} w^{-1}(x_{kn}) w|\right\|_\infty.$$

Then we have using the above inequalities and [12],

$$\begin{aligned}
&\left|L_n^{(j)}[f](x) - f^{(j)}(x)\right| \phi^j_{\frac{a_n}{n}}(x) w(x) \\
&\lesssim \left|L_n^{(j)}[f](x) - P_{n-1}^{(j)}(x)\right| \phi^j_{\frac{a_n}{n}}(x) w(x) + \left|P_{n-1}^{(j)}(x) - f^{(j)}(x)\right| \phi^j_{\frac{a_n}{n}}(x) w(x) \\
&\lesssim \left(\frac{n}{a_n}\right)^j \| |L_n[f - P_{n-1}](x)| \, w(x)\|_\infty + \left|P_{n-1}^{(j)}(x) - f^{(j)}(x)\right| \phi^j_{\frac{a_n}{n}}(x) w(x) \\
&\lesssim \left(\frac{n}{a_n}\right)^j \|L_n\|_\infty E_{n-1}[f]_{w,\infty} + T^{1/4}(a_n) E_{n-j-1}[f^{(j)}]_{w,\infty} \\
&\lesssim \|L_n\|_\infty E_{n-1-j}[f^{(j)}]_{w,\infty} + T^{1/4}(a_n) E_{n-j-1}[f^{(j)}]_{w,\infty} \\
&= \left(\|L_n\|_\infty + T^{1/4}(a_n)\right) E_{n-j-1}[f^{(j)}]_{w,\infty}.
\end{aligned}$$

This completes the proof of Theorem 2.3. □

Proof of Corollaries 2.5, 2.8, and 2.9. From ([2, 3, 19]), we know that uniformly for $n \geq N_0$,

$$\|L_n\|_{L_\infty(I)} \sim n^{1/6} T^{1/6}(a_n) \tag{3.10}$$

and from Corollary 1.4 and 1.7 in [4], Corollary 1.5 and 1.8 in [8], and Corollary 1.4 and 2.6 in II of [14], or (a) and (b) of Theorem 3.1, we have

$$\begin{aligned}
E_{n-1-j}[f^{(j)}]_{w,\infty} &\lesssim \left(\frac{a_n}{n}\right)^k \left\|f^{(j+k)} \phi^k_{\frac{a_n}{n}} w\right\|_{L_\infty(I)} \\
&\lesssim \left(\frac{a_n}{n}\right)^k \left\|f^{(j+k)} w\right\|_{L_\infty(I)} \lesssim \left(\frac{a_n}{n}\right)^k.
\end{aligned} \tag{3.11}$$

Therefore, we have the results from (3.10), (3.11), and Theorem 2.3. Specially, for the case of $w = w_\alpha$, $\alpha > 1$ given by Example 2.1, $a_n \sim n^{\frac{1}{\alpha}}$ and $T(a_n) \sim 1$ imply Corollary 2.5. □

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