Interpolatory Product Quadratures for Cauchy Principal Value Integrals with Freud Weights

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Abstract

We prove convergence results and error estimates for interpolatory product quadrature formulas for Cauchy principal value integrals on the real line with Freud–type weight functions. The formulas are based on polynomial interpolation at the zeros of orthogonal polynomials associated with the weight function under consideration. As a by–product, we obtain new bounds for the derivative of the functions of the second kind for these weight functions.

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1 Introduction and Statement of Results

The problem of the numerical evaluation of (weighted) Cauchy principal value integrals of the form

$$⨍_a^b W^2(t)\frac{f(t)}{t-x}dt = \lim_{\varepsilon\to 0+}\left(\int_a^{x-\varepsilon} W^2(t)\frac{f(t)}{t-x}dt + \int_{x+\varepsilon}^b W^2(t)\frac{f(t)}{t-x}dt\right)$$

where $a < x < b$ has recently attracted a lot of attention, see, e.g., [2,6,7,14] and the references cited therein. The main reason for this interest is probably due to the fact that integral equations with Cauchy principal value integrals have shown to be an adequate tool for the modelling of many physical situations. However, only very little is known in the case that the interval of integration is infinite. In the case of ordinary integrals (without strong singularities), results concerning infinite intervals may be found, e.g., in [12,15]. It is the aim of this paper to investigate a class of quadrature formulae for this problem and, in particular, to derive error estimates.

Therefore, we consider the problem of the numerical approximation of the Cauchy principal value integral

$$I[f;x] := ⨍_{-\infty}^{\infty} W^2(t)\frac{f(t)}{t-x}dt = \lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{x-\varepsilon} W^2(t)\frac{f(t)}{t-x}dt + \int_{x+\varepsilon}^{\infty} W^2(t)\frac{f(t)}{t-x}dt\right) \tag{1.1}$$

*Present address: Institut für Angewandte Mathematik, Technische Universität Braunschweig, Pockelsstr. 14, D-38126 Braunschweig, Germany, e-mail: K.Diethelm@tu-bs.de where $x \in \mathbb{R}$ and W is a fixed weight function. Frequently, the function $I[f;\cdot]$ is called the *Hilbert transform* of $W^2 f$. We assume the weight function W to be of Freud's type, i.e.

$$W := \exp(-Q)$$

where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is called the external field with respect to W and is of smooth polynomial growth at infinity.

More precisely, we consider the following class of Freud weights from [10].

Definition 1.1. *Let $W := \exp(-Q)$, where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuous in $\mathbb{R}$, Q'' is continuous in $(0,\infty)$, $Q' > 0$ in $(0,\infty)$, $Q(0) = 0$ and for some $A, B > 1$,*

$$A \leq \frac{d}{dt}\left(tQ'(t)\right)/Q'(t) \leq B,\ t \in (0,\infty).$$

Then we write $W \in \mathcal{F}$.

The archetypal example of such weights is

$$W_\beta(t) := \exp\left(-\frac{1}{2}|t|^\beta\right),\ t \in \mathbb{R},\ \beta > 1 \tag{1.2}$$

where $A = B = \beta$ of which the Hermite weight is a special example for $\beta = 2$.

We remark that unlike in [8], we need conditions on Q'' as at present these are the weakest conditions under which estimates for the requisite Christoffel functions generated by W are available [11]. Moreover, we need $xQ'(x)$ to be strictly increasing in $(0,\infty)$ in order to ensure the existence of the *Rakhmanov–Mhaskar–Saff* number a_u which is the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{\sqrt{1-t^2}}dt.$$

For example, for W_β, we have $a_u = C(\beta)u^{\frac{1}{\beta}}$, $u > 0$. One of its uses is the identity [3,8,11]

$$\|PW\|_{L_\infty[-a_n,a_n]} = \|PW\|_{L_\infty(\mathbb{R})},\ P \in \mathcal{P}_n,\ n \geq 1.$$

Here and in the sequel, $\mathcal{P}_n$ will denote the class of polynomials of degree at most n and $C, C_1, C_2 > 0$ will denote constants independent of n, x and $P \in \mathcal{P}_n$ that might take on different values at different occurrences. We shall write $C \neq C(L)$ to mean that the constant C is independent of L.

Finally, for real sequences α_n and $\beta_n \neq 0$, $\alpha_n = O(\beta_n)$, $\alpha_n \sim \beta_n$ and $\alpha_n = o(\beta_n)$ will mean respectively that there exists $C_j > 0$, $j = 1,2,3$ independent of n such that $\alpha_n \leq C_1\beta_n$, $C_2 \leq \frac{\alpha_n}{\beta_n} \leq C_3$ and

$$\frac{\alpha_n}{\beta_n} \longrightarrow 0,\ n \longrightarrow \infty.$$

The quadrature schemes we investigate are so–called *interpolatory product methods*, given by the following approach: Let

$$p_n\left(W^2, x\right) := \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_n\left(W^2\right) > 0$$

denote the nth orthonormal polynomial for W^2 so that

$$\int_{\mathbb{R}} p_n(x)p_m(x)W^2(x)dx = \delta_{mn}. \tag{1.3}$$

Let $x_{j,n}$, $j = 1, 2, \ldots, n$, denote its n simple zeros ordered as

$$x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n}.$$

Then it is well known [11] that for large enough n, there exists $D \neq D(n) > 0$ such that

$$\left| \frac{x_{1,n}}{a_n} - 1 \right| \leq D n^{-\frac{2}{3}}. \tag{1.4}$$

Now fix D in (1.4).

The zeros of p_n above will serve as the nodes of our quadrature formula Q_n which is therefore defined by

$$Q_n[f;x] = \sum_{j=1}^{n} w_{j,n}(x) f(x_{j,n})$$

and where the weights $w_{j,n}$ are chosen such that the quadrature error R_n satisfies

$$R_n[f;x] := I[f;x] - Q_n[f;x] = 0 \tag{1.5}$$

for every x and every $f \in \mathcal{P}_{n-1}$. (For ease of notation, we have suppressed the reference to the weight function involved). In other words,

$$Q_n[f;x] = I\left[L_n[f];x\right],$$

where $L_n[f]$ is the Lagrange interpolating polynomial for the function f with nodes $x_{j,n}$.

It might seem strange to have W^2 in our definition for p_n and not W. Recall, however, that we weight each p_n by W in our definition (1.3). This will also prove useful later in the formulation and proofs of our theorems.

Our main result is the following theorem that we shall prove in §4. Throughout,

$$E_n[f]_{W,\infty} := \inf_{P \in \mathcal{P}_n} \|(f-P)W\|_{L_\infty(\mathbb{R})} \tag{1.6}$$

denotes the error of the best weighted uniform approximation for the function f from $\mathcal{P}_n$. It is well known that for $W \in \mathcal{F}$, $E_n[f]_{W,\infty} \longrightarrow 0$, $n \longrightarrow \infty$ and that its rate can be characterised in terms of the smoothness properties of the underlying f [4,8].

To formulate the result, we introduce one more definition. We say that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies a weighted Lipschitz condition of order α with respect to the weight function W, or for short, $f \in Lip(\alpha, W)$, if

$$\sup_{x \in \mathbb{R}} \left(W(x) \sup_{x_1, x_2 \in [x-1, x+1], x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^{\alpha}} \right) < \infty \tag{1.7}$$

Definition 1.2. *Let $W \in \mathcal{F}$. We shall say that a sequence $(\delta_n)_{n=1}^{\infty}$ is an admissible sequence for the given weight W if $\delta_n > 0$ for all n and if there exists $0 < c < 1$ such that for all n sufficiently large,*

$$\delta_n < c a_n, \tag{1.8}$$

where a_n is the Rakhmanov-Mhaskar-Saff number for W.

The following is our main result.

Theorem 1.3. *Let $W \in \mathcal{F}$ and $(\delta_n)_{n=1}^{\infty}$ an admissible sequence for the given W. Let $x \in \mathbb{R}$, $0 < \eta < 1$, n large enough and let $f \in Lip(\alpha, W)$ for some $\alpha > 0$. Then for the interpolatory product quadrature formula Q_n, there exists a positive constant C_1, independent of f, n and x, such that*

$$|R_n[f,x]| \leq C_1 \left[\ln n + \gamma_n(x)\right] E_{n-1}[f]_{W,\infty} \tag{1.9}$$

where

$$\gamma_n(x) := \begin{cases} \frac{a_n}{\delta_n} & , \text{ if } |x| \geq 2a_n. \\ \frac{a_n}{\delta_n} + \delta_n n^{\frac{5}{6}} a_n^{-1} & , \text{ if } a_n \left(1 + Dn^{-\frac{2}{3}}\right) < |x| < 2a_n, \ \delta_n < 2Da_n n^{-\frac{2}{3}}. \\ \frac{a_n}{\delta_n} + \delta_n n^{\frac{7}{6}} a_n^{-1} & , \text{ if } a_n \left(1 + Dn^{-\frac{2}{3}}\right) < |x| < 2a_n, \ \text{otherwise.} \\ \frac{a_n}{\delta_n} + \delta_n n^{\frac{7}{6}} a_n^{-1} & , \text{ if } \eta a_n < |x| \leq a_n \left(1 + Dn^{-\frac{2}{3}}\right). \\ \frac{a_n}{\delta_n} + \delta_n n^{\frac{7}{6}} a_n^{-1} \ln n & , \text{ if } 0 \leq |x| \leq \eta a_n. \end{cases} \tag{1.10}$$

Note that we have some freedom in the choice of the sequence δ_n. If we are interested in error bounds on some particular subinterval of $\mathbb{R}$, we can use this freedom to derive sharper bounds on those subintervals. But we can also find a sequence that gives a uniform bound over the entire real line:

Corollary 1.4. *Under the assumptions of Theorem 1.3, there is a constant C_2 independent of f, n, and x such that*

$$|R_n[f, x]| \leq C_2 n^{7/12} \sqrt{\ln n} E_{n-1}[f]_{W,\infty}.$$

Jackson Theorems for Freud weights

The possibility of obtaining bounds for $E_{n-1}[f]_{W,\infty}$ and hence error estimates for (1.9) arises from the correct Jackson/Bernstein estimates proved in [8] and further explored in [4]. We will need:

Theorem 1.5. *Let $W \in \mathcal{F}$, $r \geq 1$, and $fW \in L_\infty(\mathbb{R})$. Furthermore, let f be continuous and suppose $\lim_{|x|\to\infty} f(x)W(x) = 0$. Then for large enough n, there exist $C_j > 0$, $j = 1, 2$ independent of f and n such that*

$$\begin{aligned} E_n[f]_{W,\infty} &\leq C_1 \omega_{r,\infty}(f, W, a_n/n) \\ &\leq C_2 a_n^r n^{-r} \left\| f^{(r)} W \right\|_{L_\infty(\mathbb{R})} \end{aligned} \tag{1.11}$$

provided that the expression on the right–hand side of (1.11) exists.

Here, the modulus of continuity $\omega_{r,\infty}(f, W, t)$ is given by (see [4,8])

$$\omega_{r,\infty}(f, W, t) = \sup_{0<h\leq t} \|W \Delta_h^r(f, \cdot)\|_{L_\infty(-\sigma(h),\sigma(h))} + \inf_{p\in\mathcal{P}_{r-1}} \|(f - P)W\|_{L_\infty(|x|\geq\sigma(t))}$$

where

$$\sigma(t) := \inf \left\{ a_n : \frac{a_n}{n} \leq t \right\},$$

$t > 0$ but is typically small and Δ_h^r is the usual rth order symmetric difference operator with stepsize h.

We note that the definition of $Lip(\alpha, W)$ and results of [4] imply for every fixed $\alpha > 0$ and $r \geq 1$,

$$f \in Lip(\alpha, W) \Longrightarrow \omega_{r,\infty}(f, W, t) = O\left(\omega_{1,\infty}(f, W, t)\right) = O(t^\alpha).$$

See [4] for related results.

As a consequence of Theorems 1.3 and 1.5, we obtain

Corollary 1.6. *Let $W \in \mathcal{F}$, δ_n an admissible sequence for W, $x \in \mathbb{R}$, $r \geq 1$, $\alpha > 0$, n large enough and $f \in Lip(\alpha, W)$. Suppose further that $\lim_{|x|\to\infty} f(x)W(x) = 0$ and that $\left\| f^{(r)} W \right\|_{L_\infty(\mathbb{R})} < \infty$.*

(a) *Then there exists* $C \neq C(f,n,x) > 0$ *such that,*

$$|R_n[f,x]| \leq C \|\gamma_n\|_{L_\infty(\mathbb{R})} a_n^r n^{-r} \left\| f^{(r)} W \right\|_{L_\infty(\mathbb{R})} \tag{1.12}$$

where γ_n *is as in (1.10).*

(b) *In particular if* $\beta > 1$, *there exists* $C_1 \neq C_1(f,n,x) > 0$ *such that*

$$|R_n[f,x]| \leq C_1 \|\gamma_n\|_{L_\infty(\mathbb{R})} n^{-r(1-1/\beta)} \left\| f^{(r)} W_\beta \right\|_{L_\infty(\mathbb{R})}. \tag{1.13}$$

Remark 1.7.

(a) We remark that often for practical applications, it is important that the error $R_n[f, .]$ goes to zero for the given admissible sequence δ_n for a large class of functions f. Since the continuity of f is not sufficient to ensure that the integral $I[f;x]$ exists, we take a closer look at a slightly smaller class of functions, namely those C^1 functions that satisfy that $f'W \in L_\infty(\mathbb{R})$. In view of (1.11) this seems to be a rather natural choice. Indeed suppose further that δ_n satisfies:

$$\delta_n^{-1} = o\left(\frac{n}{a_n^2}\right), \; n \longrightarrow \infty \tag{1.14a}$$

and

$$\delta_n n^{\frac{1}{6}} (\ln n)^{1+\delta} = o(1), \; n \longrightarrow \infty \tag{1.14b}$$

for some $\delta > 0$. Then (1.9), (1.10) and Theorem 1.5, show that for a given fixed $x \in \mathbb{R}$,

$$R_n[f,x] \longrightarrow 0, \; n \longrightarrow \infty$$

with a rate depending on n, a_n, δ_n and the location of x. Given a particular weight function W, it is easy to construct examples of such sequences δ_n.

(b) We now observe that by considering the weight W_β of (1.2) it is easy to see that for $\beta < \frac{12}{5}$ the conditions (1.14a) and (1.14b) are no longer compatible. Indeed for $\beta \in \left(1, \frac{12}{5}\right)$ we are not sure if indeed the error $R_n[f, .]$ does go to zero for large enough n and all the functions considered in (a). We leave this as an open problem. However, we can of course say that for functions with better smoothness properties, Theorem 1.5 yields that the factor $E_{n-1}[f]_{W,\infty}$ decreases faster, and therefore we may allow a faster growth rate for the factor $[\ln n + \gamma_n(x)]$ in (1.9), i.e. we can replace conditions (1.14a) and (1.14b) by weaker conditions without losing the overall convergence statement. Therefore, if the function f is sufficiently smooth, we can still expect convergence even in cases where the weight function does not satisfy the restriction mentioned above.

It turns out that the functions of the second kind q_n associated with the orthogonal polynomials for W^2, defined by

$$q_n(x) := \int_{-\infty}^{\infty} W^2(t) \frac{p_n(t)}{t-x} dt, \tag{1.15}$$

play an important role in the proof of Theorem 1.3. These functions are also of interest in their own right, cf. [1,5]. Until recently, not much has been known about the derivatives of these functions. In §2, we shall prove a pointwise bound for these derivatives in the following sense:

Theorem 1.8. *Let* $W \in \mathcal{F}$, $x \in \mathbb{R}$, $0 < \eta < 1$ *and* n *large enough. Then there exists* $C \neq C(n,x) > 0$ *such that*

$$|q_n'(x)| \leq \begin{cases} Cn^{7/6} a_n^{-3/2} \ln n & , \; \textit{if } |x| \leq \eta a_n \\ Cn a_n^{-\frac{3}{2}} & , \; \textit{if } |x| > \eta a_n \end{cases} \tag{1.16}$$

Our final theorem gives an error estimate that is independent of the choice of the nodes of the quadrature formula. Therefore, it is of interest for other quadrature formulae too. Moreover, it holds for a much larger class of weights than $\mathcal{F}$.

Theorem 1.9 *Let W be a bounded weight function on $\mathbb{R}$ with finite moments and with the property $W^2 \in Lip\left(\beta, W^{-1}\right)$ for some $\beta > 0$. Let $x \in \mathbb{R}$ and $f \in Lip(\alpha, W)$ for some $\alpha > 0$. Furthermore, let $P_{n-1}^* \in \mathcal{P}_{n-1}$ be the polynomial of best uniform approximation to f with respect to the weight function W, satisfying*

$$\left\|W\left(f-P_{n-1}^*\right)\right\|_{L_\infty(\mathbb{R})} = \inf_{P \in \mathcal{P}_{n-1}} \|(f-P)W\|_{L_\infty(\mathbb{R})} = E_{n-1}[f]_{W,\infty}.$$

Then,

$$\left|I[f-P_{n-1}^*;x]\right| \leq C E_{n-1}[f]_{W,\infty} \ln n, \tag{1.17}$$

where C is a constant depending on W, but independent of f, n, and x.

This paper is organised as follows. In Section 2, we present the proof of Theorem 1.8. In Section 3, we present an important auxiliary result and the proof of Theorem 1.9 which is of independent interest. Then, in Section 4, we present the proof of Theorem 1.3 and Corollaries 1.4 and 1.6. Finally, Section 5 contains some numerical examples.

2 Proof of Theorem 1.8

We begin with two technical lemmas. The first gives some properties of a_u, p_n and the spacing of the zeros of p_n.

Lemma 2.1. *Let $W \in \mathcal{F}$ and define the sequence of functions*

$$\psi_n(x) := \max\left\{n^{-\frac{2}{3}}, 1-\frac{|x|}{a_n}\right\}, \ x \in \mathbb{R}, \ n \geq 1. \tag{2.1}$$

(a) Uniformly for $n \geq 1$ and $1 \leq j \leq n-1$, there holds

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n}\psi_n\left(x_{j,n}\right)^{-\frac{1}{2}} \tag{2.2}$$

(b) Uniformly for $1 \leq j \leq n$ and $t \in [x_{j+1,n}, x_{j,n}]$, there holds

$$\psi_n(t) \sim \psi_n\left(x_{j,n}\right). \tag{2.3}$$

(c) a_u is strictly increasing with u and for $u \in [1,\infty)$

$$u^{\frac{1}{B}} \leq \frac{a_u}{a_1} \leq u^{\frac{1}{A}}. \tag{2.4}$$

(d) Let $0 < a < b < \infty$. Then, for the function Q in Definition 1.1 we have, uniformly for $x \in [a,b]$ and $n \geq N_0$,

$$a_n Q'\left(a_n x\right) \sim Q\left(a_n x\right) \sim n. \tag{2.5}$$

Proof. (2.2), (2.3), (2.4) and (2.5) are respectively (4.17) of [1], (2.8) of [13], Lemma 2.2 (a) of [8] and (4.23) and (4.3) of [1]. □

The next lemma gives L_p bounds on p_n, q_n and some Markov–Bernstein inequalities.

Lemma 2.2. *Let $W \in \mathcal{F}$.*

(a) *Uniformly for* $n \geq 1$,

$$\|p_n W\|_{L_\infty(\mathbb{R})} \sim a_n^{-\frac{1}{2}} n^{\frac{1}{6}}. \tag{2.6}$$

(b) *Let* $0 < p < \infty$. *Then uniformly for* $n \geq 1$,

$$\|p_n W\|_{L_p(\mathbb{R})} \sim a_n^{\frac{1}{p}-\frac{1}{2}} \begin{cases} 1 & ,p < 4 \\ (\ln n)^{\frac{1}{4}} & ,p = 4 \\ n^{\frac{1}{6}\left(1-\frac{p}{4}\right)} & ,p > 4 \end{cases} . \tag{2.7}$$

(c) *Let* $1 < p \leq \infty$. *Then, uniformly for* $n \geq 1$,

$$\|q_n\|_{L_p(\mathbb{R})} \sim a_n^{-\frac{1}{2}}. \tag{2.8}$$

(d) *Let* $0 < p \leq \infty$, n *large enough and* $P \in \mathcal{P}_n$ *.Then there exists* $C \neq C(n, P) > 0$ *such that*

$$\left\|\left(PW^2\right)'\right\|_{L_p(\mathbb{R})} \leq C\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})} \tag{2.9}$$

Proof. (2.6) is (4.5) in [11], (2.7) is (4.8) in [13] and (2.8) is (2.5) in [1]. We prove (2.9). Suppose first that $p = \infty$. Then (2.9) follows easily from the Markov–Bernstein inequality

$$\|(PW)'\|_{L_\infty(\mathbb{R})} \leq C\frac{n}{a_n}\|PW\|_{L_\infty(\mathbb{R})} \tag{2.10}$$

(see for example (4.15) in [1]), recalling that W decays exponentially. For $0 < p < \infty$, we use (1.5) of [10]:

$$\left\|(PW)'\psi_n^{\frac{-1}{2}}\right\|_{L_p(\mathbb{R})} \leq C_1\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})}.$$

Then observing that for $|x| \leq \eta a_n$ for any fixed $\eta > 0$, $\psi_n^{\frac{1}{2}}(x) = O(1)$ and for other $|x|$, $\psi_n^{\frac{1}{2}}(x)$ grows at most as fast as a power of $|x|^{\frac{1}{2}}$ easily gives using again the exponential decay of W, (2.4), (2.5) and the Markov–Bernstein inequality above,

$$\begin{aligned} \left\|\left(PW^2\right)'\right\|_{L_p(\mathbb{R})} &= \left\|\left(PW^2\right)'\psi_n^{\frac{-1}{2}}\psi_n^{\frac{1}{2}}\right\|_{L_p(\mathbb{R})} \\ &\leq C_2\left\|(PW)'\psi_n^{\frac{-1}{2}}\right\|_{L_p(\mathbb{R})} \leq C_3\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})}. \ \square \end{aligned}$$

We are ready for the

Proof of Theorem 1.8. We introduce the abbreviation

$$\rho_n(x) := W^2(x)p_n(x)$$

and consider the case $|x| \leq \eta a_n$ first. Under our assumptions on W, we may integrate partially in (1.15). Then, we can see that the exchange of differentiation and integration in the resulting expression is legal if we interpret the resulting integral in the Cauchy principal value sense. Hence, we obtain

$$q_n'(x) = \int_{-\infty}^{\infty} \rho_n'(t)\frac{1}{t-x}dt.$$

Defining $\epsilon_n := a_n/n$, we write

$$q_n'(x) = I_1 + I_2,$$

where

$$I_1 = \int_{|t-x|\geq\epsilon_n} \rho_n'(t)\frac{1}{t-x}dt$$

and

$$I_2 = ⨍_{|t-x|<\epsilon_n} \rho_n'(t)\frac{1}{t-x}dt = \int_{|t-x|<\epsilon_n} \frac{\rho_n'(t)-\rho_n'(x)}{t-x}dt.$$

First, we deal with I_1. We apply Hölder's inequality and derive

$$\left|\int_{|t-x|\geq\epsilon_n} \frac{\rho_n'(t)}{t-x}dt\right| \leq \left\|\rho_n' W^{-1/2}\right\|_{L_\infty(\mathbb{R})} \left\|(\,\cdot\,-x)^{-1}W^{1/2}\right\|_{L_1(S_n)} \tag{2.11}$$

where $S_n = \{t \in \mathbb{R} : \ |t-x| \geq \epsilon_n\}$. An explicit calculation gives

$$\left\|(\,\cdot\,-x)^{-1}W^{1/2}\right\|_{L_1(S_n)} \sim |\ln\epsilon_n| \sim \ln n, \tag{2.12}$$

uniformly in n and so by an application of the L_∞ Markov–Bernstein inequality (2.10) and (2.7), we deduce

$$\begin{aligned}
\left\|\rho_n' W^{-1/2}\right\|_{L_\infty(\mathbb{R})} &= \left\|(W^2p_n)'W^{-1/2}\right\|_{L_\infty(\mathbb{R})} \\
&= \left\|W^{1/2}[(Wp_n)' + W'p_n]\right\|_{L_\infty(\mathbb{R})} \\
&\leq \left\|W^{1/2}W'p_n\right\|_{L_\infty(\mathbb{R})} + \left\|W^{1/2}\right\|_{L_\infty(\mathbb{R})} \|(Wp_n)'\|_{L_\infty(\mathbb{R})} \\
&\leq \left\|W^{1/2}Q'Wp_n\right\|_{L_\infty(\mathbb{R})} + C\frac{n}{a_n}\|Wp_n\|_{L_\infty(\mathbb{R})} \\
&\leq \left\|W^{1/2}Q'\right\|_{L_\infty(\mathbb{R})}\|Wp_n\|_{L_\infty(\mathbb{R})} + C\frac{n}{a_n}\|Wp_n\|_{L_\infty(\mathbb{R})} \\
&\leq C\frac{n}{a_n}\|Wp_n\|_{L_\infty(\mathbb{R})} \leq C_1\frac{n}{a_n}a_n^{-1/2}n^{1/6}.
\end{aligned} \tag{2.13}$$

Thus combining our estimates (2.12) and (2.13) and substituting into (2.11) readily gives

$$|I_1| \leq C_2\frac{n^{7/6}}{a_n^{3/2}}\ln n. \tag{2.14}$$

For I_2, we observe that

$$|I_2| \leq 2\epsilon_n\|\rho_n''\|_{L_\infty(\mathbb{R})}.$$

Now applying the L_∞ Markov–Bernstein inequality (2.10) twice, we obtain

$$\|\rho_n''\|_{L_\infty(\mathbb{R})} \leq C_3n^2a_n^{-2}\|Wp_n\|_{L_\infty(\mathbb{R})} \leq C_4n^{13/6}a_n^{-5/2},$$

and thus

$$|I_2| \leq C_5n^{7/6}a_n^{-3/2}. \tag{2.15}$$

Combining our bounds (2.14) and (2.15) for I_1 and I_2, gives (1.16) for this case. Notice that the above estimate works for every $x \in \mathbb{R}$; however, we observe that estimating differently for the range $|x| > \eta a_n$ using the method of [1], gives a better estimate for this range of x. We proceed henceforth and write for the new range of x

$$\begin{aligned}
q_n'(x) &= \int_{|t-x|\geq\frac{\eta a_n}{2}} + \int_{n^{-10}<|t-x|<\frac{\eta a_n}{2}} + ⨍_{|t-x|\leq n^{-10}} \frac{\rho_n'(t)}{t-x}dt \\
&= I_3 + I_4 + I_5.
\end{aligned} \tag{2.16}$$

We begin with the estimation of I_5. Note that using (2.10) and (2.6) gives

$$|I_5| = \left|\int_{x-n^{-10}}^{x+n^{-10}} \frac{\rho_n'(t)-\rho_n'(x)}{t-x}dt\right| \leq C_6 n^{-10}\,\|\rho_n''\|_{L_\infty(\mathbb{R})}$$

$$\leq C_7\left(\frac{n}{a_n}\right)^2 n^{-10}\,\|p_n W\|_{L_\infty(\mathbb{R})} \leq C_8 n^{-\frac{47}{6}} a_n^{-\frac{5}{2}}. \tag{2.17}$$

Next we estimate I_4. We have

$$\begin{aligned} |I_4| &\leq \int_{n^{-10}<|t-x|<\frac{\eta a_n}{2}} \frac{dt}{|t-x|}\|\rho_n'\|_{L_\infty\left(n^{-10}<|t-x|<\frac{\eta a_n}{2}\right)} \\ &\leq C_9\|\rho_n'\|_{L_\infty\left(n^{-10}<|t-x|<\frac{\eta a_n}{2}\right)} \ln n. \end{aligned} \tag{2.18}$$

Recalling that $|x|>\eta a_n$ we may write

$$\begin{aligned} &\|\rho_n'\|_{L_\infty\left(n^{-10}<|t-x|<\frac{\eta a_n}{2}\right)} \\ &= \|W[p_n'W-2p_nQ'W]\|_{L_\infty\left(n^{-10}<|t-x|<\frac{\eta a_n}{2}\right)} \\ &\leq C_{10}W\left(\frac{\eta a_n}{2}\right)\left(\|p_n'W\|_{L_\infty(\mathbb{R})}+\left|Q'\left(\frac{3\eta a_n}{2}\right)\right|\|p_nW\|_{L_\infty(\mathbb{R})}\right) \\ &\leq C_{11}a_n^{-\frac{1}{2}} \end{aligned}$$

recalling (2.6), (2.10), (2.5) and the fact that (since a_n increases polynomially in n), $W\left(\frac{\eta a_n}{2}\right)$ decays exponentially, whereas $Q'(\frac{3\eta a_n}{2})$ grows at most as fast as a polynomial. Thus (2.18) becomes

$$|I_4| \leq C_{11}a_n^{\frac{-1}{2}}\ln n. \tag{2.19}$$

Finally using (2.7) and (2.9) gives

$$\begin{aligned} |I_3| &\leq \frac{2}{\eta a_n}\int_{|t-x|\geq\frac{\eta a_n}{2}} |\rho_n'(t)|\,dt \leq \frac{2}{\eta a_n}\,\|\rho_n'\|_{L_1(\mathbb{R})} \\ &\leq C_{12}\frac{n}{a_n^{\frac{3}{2}}}. \end{aligned} \tag{2.20}$$

Combining our estimates (2.17), (2.19) and (2.20) and taking into consideration that $a_n \geq Cn^{1/B}$ for some $B>1$ (cf. (2.4) and Definition 1.1) gives (1.16) for this range of x. $\square$

3 Proof of Theorem 1.9 and an Auxiliary Result

In this section, we prove Theorem 1.9 and derive a lemma which is useful in the proof of Theorem 1.3. We begin with:

Lemma 3.1. *For the weights $w_{j,n}$ of the quadrature formula Q_n defined in (1.5), we have*

$$w_{j,n}(x) = \begin{cases} \frac{q_n(x_{j,n})-q_n(x)}{(x_{j,n}-x)p_n'(x_{j,n})} & \text{if } x\neq x_{j,n} \\ \frac{q_n'(x_{j,n})}{p_n'(x_{j,n})} & \text{if } x = x_{j,n} \end{cases}. \tag{3.1}$$

Proof. Obviously,

$$w_{k,n}(x) = I\left[l_{k,n};x\right], \tag{3.2}$$

where $l_{k,n}$ is the kth fundamental polynomial of the Lagrange interpolation process associated with the $x_{j,n}$. Using the well known explicit representation of $l_{k,n}$ and the definition of the function of the second kind, the result follows immediately. $\square$

We now present the proof of Theorem 1.9. In the case of a finite interval of integration, similar results are known, see Lemma 3.4 of [2]. However, the fact that we want a uniform bound over an infinite range (note that the constant C does not depend on x) prevents a direct generalization of the proof that works in the finite–range case.

Proof of Theorem 1.9. Let $0 < \epsilon < 1$. Then, we see that

$$\left|\int_{-\infty}^{x-\epsilon} W^2(t)\frac{f(t)-P_{n-1}^*(t)}{t-x}dt\right| \le \|W(f-P_{n-1}^*)\|_{L_\infty(\mathbb{R})}\left|\int_{-\infty}^{x-\epsilon} W(t)\frac{1}{t-x}dt\right|.$$

Noting that

$$\begin{aligned}\left|\int_{-\infty}^{x-\epsilon} W(t)\frac{1}{t-x}dt\right| &\le \int_{-\infty}^{x-1} W(t)\frac{1}{x-t}dt + \int_{x-1}^{x-\epsilon} W(t)\frac{1}{x-t}dt \\ &\le \int_{-\infty}^{x-1} W(t)dt + \|W\|_{L_\infty[x-1,x]}\int_{x-1}^{x-\epsilon}\frac{1}{x-t}dt \\ &= \int_{-\infty}^{x-1} W(t)dt + \|W\|_{L_\infty[x-1,x]}\ln\epsilon^{-1},\end{aligned}$$

we derive

$$\left|\int_{-\infty}^{x-\epsilon} W^2(t)\frac{f(t)-P_{n-1}^*(t)}{t-x}dt\right| \le E_{n-1}[f]_{W,\infty}\left(\int_{-\infty}^{x-1} W(t)dt + \|W\|_{L_\infty[x-1,x]}\ln\epsilon^{-1}\right).$$

The integral over $(x+\epsilon,\infty)$ can be bounded in a similar way, so we obtain

$$\left|\int_{|t-x|\ge\epsilon} W^2(t)\frac{f(t)-P_{n-1}^*(t)}{t-x}dt\right| \le E_{n-1}[f]_{W,\infty}\left(\|W\|_{L_1(\mathbb{R})} + 2\|W\|_{L_\infty[x-1,x+1]}\ln\epsilon^{-1}\right). \quad (3.3)$$

For the integral over the remaining subinterval, our method is similar to that of [7]. See also [6, Proof of Theorem 3.1]). We begin by writing

$$\begin{aligned}&⨍_{x-\epsilon}^{x+\epsilon} W^2(t)\frac{f(t)-P_{n-1}^*(t)}{t-x}dt \\ &\quad= \int_{x-\epsilon}^{x+\epsilon} W^2(t)\frac{\left(f(t)-P_{n-1}^*(t)\right)-\left(f(x)-P_{n-1}^*(x)\right)}{t-x}dt \\ &\qquad + W(x)\left(f(x)-P_{n-1}^*(x)\right)⨍_{x-\epsilon}^{x+\epsilon}\frac{W^2(t)}{W(x)}\,\frac{dt}{t-x}\end{aligned}$$

Now, by a method similar to that of Kalandiya [9, pp.105–107], we may see that, because of the weighted Lipschitz condition on f,

$$W(t)\frac{\left|\left(f(t)-P_{n-1}^*(t)\right)-\left(f(x)-P_{n-1}^*(x)\right)\right|}{|t-x|^{\alpha^*}} \le C\left(\frac{n}{a_n}\right)^{2\alpha^*}E_{n-1}[f]_{W,\infty}$$

for some $\alpha^* \in (0,\alpha/2)$. Thus,

$$\left|\int_{x-\epsilon}^{x+\epsilon} W^2(t)\frac{\left(f(t)-P_{n-1}^*(t)\right)-\left(f(x)-P_{n-1}^*(x)\right)}{t-x}dt\right|$$

$$
\begin{aligned}
&\le \|W\|_{L_\infty[x-1,x+1]} \int_{x-\epsilon}^{x+\epsilon} W(t) \frac{\left|\left(f(t)-P_{n-1}^*(t)\right)-\left(f(x)-P_{n-1}^*(x)\right)\right|}{|t-x|^{\alpha^*}} \frac{dt}{|t-x|^{1-\alpha^*}} \\
&\le C\left(\frac{n}{a_n}\right)^{2\alpha^*} \|W\|_{L_\infty[x-1,x+1]} E_{n-1}[f]_{W,\infty} \int_0^\epsilon y^{\alpha^*-1} dy \\
&= C\left(\frac{n}{a_n}\right)^{2\alpha^*} \|W\|_{L_\infty[x-1,x+1]} E_{n-1}[f]_{W,\infty} \epsilon^{\alpha^*}. \qquad (3.4)
\end{aligned}
$$

Furthermore, using the weighted Lipschitz condition on W^2,

$$
\begin{aligned}
\left|\int_{x-\epsilon}^{x+\epsilon} \frac{W^2(t)}{W(x)} \frac{dt}{t-x}\right| &= \frac{1}{W(x)} \left|\int_{x-\epsilon}^{x+\epsilon} \frac{W^2(t)-W^2(x)}{t-x} dt\right| \\
&\le \frac{1}{W(x)} \int_{x-\epsilon}^{x+\epsilon} \frac{\left|W^2(t)-W^2(x)\right|}{|t-x|^\beta} \frac{dt}{|t-x|^{1-\beta}} \\
&\le C \int_0^\varepsilon y^{\beta-1} dy = C\varepsilon^\beta. \qquad (3.5)
\end{aligned}
$$

Combining our estimates (3.4) and (3.5), we therefore obtain

$$
\left|\int_{x-\epsilon}^{x+\epsilon} W^2(t) \frac{f(t)-P_{n-1}^*(t)}{t-x} dt\right| \le C_1 E_{n-1}[f]_{W,\infty} \left(\epsilon^\beta + \left(\frac{n}{a_n}\right)^{2\alpha^*} \|W\|_{L_\infty[x-1,x+1]} \epsilon^{\alpha^*}\right). \qquad (3.6)
$$

(1.17) and hence the theorem follows by combining (3.3) with (3.6) with $\epsilon := n^{-2}$, recalling that W is bounded and that a_n increases to infinity. □

4 Proof of Theorem 1.3 and Corollaries 1.4 and 1.6

The proof of Theorem 1.3 is based on the fact that our quadrature formula Q_n is of interpolatory type, i.e. it is exact for all polynomials of degree $\le n-1$. Thus,

$$
\begin{aligned}
|R_n[f;x]| &= \left|R_n[f-P_{n-1}^*;x]\right| \\
&\le \left|I[f-P_{n-1}^*;x]\right| + \sum_{j=1}^n \frac{|w_{j,n}(x)|}{W(x_{j,n})} \left[W(x_{j,n})\left|f(x_{j,n})-P_{n-1}^*(x_{j,n})\right|\right]. \qquad (4.1)
\end{aligned}
$$

where P_{n-1}^* is the polynomial of best uniform approximation for f from $\mathcal{P}_{n-1}$ with respect to the weight function W. A bound for the first expression on the right-hand side of (4.1) is given in Theorem 1.9 and the term in brackets is bounded by $E_{n-1}[f]_{W,\infty}$. Hence, we now have to prove that

$$
\sum_{j=1}^n \frac{|w_{j,n}(x)|}{W(x_{j,n})} \le C\gamma_n(x) \qquad (4.2)
$$

where γ_n is given by (1.10).

Before we present the proof we need a simple lemma on $p_n' W(x_{j,n})$.

Lemma 4.1. *Let $W \in \mathcal{F}$, and recall the definition of ψ_n in (2.1).*

(a) Uniformly for $1 \le j \le n$ there holds

$$
|p_n' W|^{-1}(x_{j,n}) \sim n^{-1} a_n^{\frac{3}{2}} \psi_n(x_{j,n})^{-\frac{1}{4}}. \qquad (4.3)
$$

(b) Uniformly for $1 \le j \le n$ there holds

$$
|p_n' W|^{-1}(x_{j,n}) \sim (x_{j,n}-x_{j+1,n})\, a_n^{\frac{1}{2}} \psi_n(x_{j,n})^{\frac{1}{4}}. \qquad (4.4)
$$

Proof. (4.3) is (4.11) in [1]. To prove (4.4), observe, using (2.2), that we have by (4.3)

$$\begin{aligned}|p_n'W|^{-1}(x_{j,n}) &\sim \frac{a_n}{n}\psi_n(x_{j,n})^{\frac{-1}{2}}a_n^{\frac{1}{2}}\psi_n(x_{j,n})^{\frac{1}{4}} \\ &\sim (x_{j,n}-x_{j+1,n})\,a_n^{\frac{1}{2}}\psi_n(x_{j,n})^{\frac{1}{4}}. \ \square\end{aligned}$$

Armed with (4.3) and (4.4) we proceed to bound the sum (4.2).

We consider our admissible $(\delta_n)_{n=1}^{\infty}$ which $\longrightarrow 0$, $n \longrightarrow \infty$ and thus split up the sum according to

$$\sum_{j=1}^{n}\frac{|w_{j,n}(x)|}{W(x_{j,n})} = \sum_{\substack{j=1\\ |x-x_{j,n}|>\delta_n}}^{n}\frac{|w_{j,n}(x)|}{W(x_{j,n})} + \sum_{\substack{j=1\\ |x-x_{j,n}|\le\delta_n}}^{n}\frac{|w_{j,n}(x)|}{W(x_{j,n})}. \tag{4.5}$$

Recall that the $w_{j,n}(x)$ are given by (3.1). For the first part, we use (2.3), (2.8) and (4.4) to obtain

$$\begin{aligned}\sum_{\substack{j=1\\ |x-x_{j,n}|>\delta_n}}^{n}\frac{|w_{j,n}(x)|}{W(x_{j,n})} &\le C\,\|q_n\|_{\infty}\,\delta_n^{-1}\sum_{j=1}^{n}a_n^{1/2}(x_{j+1,n}-x_{j,n})\psi_n(x_{j,n})^{1/4} \\ &\le C\delta_n^{-1}\int_{|t|\le a_n(1+C_1n^{-2/3})}\psi_n(t)^{1/4}dt \\ &\le C_1\delta_n^{-1}a_n,\end{aligned}$$

where we have used the fact that the sum in the first line is a Riemann sum for the integral in the term following it.

Looking at the second part of the sum, we consider several cases. We may assume, much as in [3] that $x \ge 0$.

Case 1: $x \ge 2a_n$.

From (1.4), we know that all the zeros of p_n are in the interval

$$\left[-a_n\left(1+Dn^{-2/3}\right), a_n\left(1+Dn^{-2/3}\right)\right].$$

Now observing that the length of the interval $\left[a_n\left(1+Dn^{-\frac{2}{3}}\right), 2a_n\right]$ is $a_n\left(1-Dn^{-\frac{2}{3}}\right)$ and recalling from (1.8) that $\delta_n < a_n$ shows that for n large enough the second sum is empty, and we have indeed that

$$\sum_{j=1}^{n}\frac{|w_{nj}(x)|}{W(x_{nj})} \le C_1a_n\delta_n^{-1} = C_1\gamma_n(x).$$

Case 2: $a_n(1+Dn^{-2/3}) \le x \le 2a_n$, $\delta_n < 2Da_nn^{-\frac{2}{3}}$.

Note that if we sum over those zeros that lie in $\left[a_n(1-Dn^{-2/3}), a_n(1+Dn^{-2/3})\right]$, (2.2) implies that the spacing between those zeros $\sim \frac{a_n}{n^{\frac{2}{3}}}$. As the length of the aforementioned interval $\sim \frac{a_n}{n^{\frac{2}{3}}}$, the number of zeros there is at most finite. Moreover, as $\delta_n < 2Da_nn^{-\frac{2}{3}}$, the zeros over which we sum can only come from this interval and so the sum is finite. Now, by the mean value theorem and Theorem 1.8 (note that we have $\xi_{j,n} > \eta a_n$)

$$\left|\frac{q_n(x)-q_n(x_{j,n})}{x-x_{j,n}}\right| = |q_n'(\xi_{j,n})| \le C_3na_n^{-3/2}.$$

Furthermore, by (4.3)

$$|p_n'(x_{j,n})W(x_{j,n})|^{-1} \le C_4a_n^{3/2}n^{-1}\psi_n(x_{j,n})^{-1/4}.$$

Using these two relations and (3.1), we find that

$$\begin{aligned}\sum_{\substack{j=1\\|x-x_{j,n}|\le\delta_n}}^{n}\frac{|w_{j,n}(x)|}{W(x_{j,n})} &\le C_5\sum_{\substack{j=1\\|x-x_{j,n}|\le\delta_n}}^{n}\psi_n(x_{j,n})^{-1/4}\\ &\le C_6\delta_n n^{2/3}a_n^{-1}\sum_{\substack{j=1\\|x-x_{j,n}|\le\delta_n}}^{n} n^{\frac{1}{6}}\\ &\le C_7 n^{5/6}\delta_n a_n^{-1}\end{aligned}$$

and so

$$\sum_{j=1}^{n}\frac{|w_{j,n}(x)|}{W(x_{j,n})}\le C_8\gamma_n(x).$$

Case 3: $a_n(1+Dn^{-2/3})\le x\le 2a_n$, otherwise.

We proceed much as in Case 2 except that we have one fundamental difference. Note that we are now allowed to sum over more zeros and at worst their spacing is $\sim\frac{a_n}{n}$. Thus we have at most $O\left(\delta_n\frac{n}{a_n}\right)$ terms in our sum. We now proceed as before.

Case 4 is exactly the same as Case 3.

Finally in Case 5, we observe that $\psi_n(x_{j,n})^{-\frac{1}{4}}\le C_9$ and proceed as before except that we must now use the first of the two bounds given in Theorem 1.8. We have completed the proof of Theorem 1.3. □

The proof of Corollary 1.4 follows easily from Theorem 1.3 upon choosing

$$\delta_n := \frac{a_n}{n^{7/12}\sqrt{\ln n}},$$

which clearly is an admissible sequence. □

Proof of Corollary 1.6. An explicit calculation easily reveals that

$$\ln n = O(\|\gamma_n\|_{L_\infty(\mathbb{R})}),$$

so (1.12) follows from (1.9) and (1.11). Moreover, (1.13) follows from (1.12) using the well-known growth property of the a_n for the weight W_β, cf. (2.4). □

5 Numerical Examples

In order to illustrate the quality of the approximation provided by our algorithm, we now state some numerical examples.

In the first example, we look at the Hermite weight function, $W^2(t)=\exp(-t^2)$, and the function $f(t)=(1+t^2)^{-1}$. The results are given in the following table.

	Exact	Error for			
x	value	$n=5$	$n=10$	$n=20$	$n=40$
0	0	0	0	0	0
0.001	−0.004888	$-3.43E-4$	$-4.75E-4$	$-1.08E-4$	$-1.13E-5$
0.01	−0.048875	$-3.43E-3$	$-4.74E-3$	$-1.08E-3$	$-1.13E-4$
0.1	−0.48165	$-3.19E-2$	$-4.51E-2$	$-9.90E-3$	$-9.71E-4$
1	−1.62537	$1.22E-1$	$3.04E-2$	$-5.08E-4$	$-1.64E-4$
2	−0.750962	$-1.39E-2$	$-2.21E-3$	$-3.21E-5$	$2.89E-5$
5	−0.272251	$3.89E-3$	$-3.18E-4$	$-8.54E-6$	$-4.80E-8$

We can see the good convergence at all points. Note in particular that the errors are very small even for those values of x that are far outside the interval bounded by the first and the last quadrature node, i. e. the interval $[-(1+Dn^{-2/3})a_n, (1+Dn^{2/3})a_n]$. The presence of the poles of f at $\pm i$ does not cause any numerical problems.

The second example is the non-symmetric function $f(t) = e^t t^4$, also with the Hermite weight $W^2(t) = \exp(-t^2)$. The results are as follows.

	Exact	Error for		
x	value	$n=5$	$n=10$	$n=20$
0	1.99139	$7.20E+0$	$2.08E-11$	$< 1E-16$
0.001	1.99310	$7.20E+0$	$8.74E-4$	$-8.37E-10$
0.01	2.00858	$7.22E+0$	$8.74E-3$	$-8.37E-9$
0.1	2.17590	$7.02E+0$	$8.50E-2$	$-7.81E-8$
1	5.18018	$-6.64E+0$	$-1.33E-1$	$-8.01E-9$
2	-3.02457	$2.66E+0$	$1.91E-2$	$-6.28E-10$
5	-1.10554	$-2.14E-2$	$-4.94E-6$	$4.80E-14$
10	-0.427988	$-2.82E-3$	$-2.18E-9$	$-9.58E-13$

Here we can see that even though f grows rapidly as $t \to \infty$, already rather small values of n give a high accuracy of the approximation.

It may be observed that we have quite rapid convergence in both examples even though the condition $\beta > 12/5$ of Remark 1.7(b) is violated. The reason for this is, as already mentioned in that remark, that the error bound stated in (1.9) contains two factors, namely $[\ln n + \gamma_n(x)]$ and $E_{n-1}[f]_{W,\infty}$. Remark 1.7(b) mainly addresses the fact that the expression in brackets (whose behaviour is essentially governed by $\gamma_n(x)$) grows as $n \to \infty$. However, for our examples the $E_{n-1}[f]_{W,\infty}$ factor goes to zero much more rapidly because of the nice smoothness properties of the functions under consideration, and therefore it is not unexpected to see that the error also decreases quickly.

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